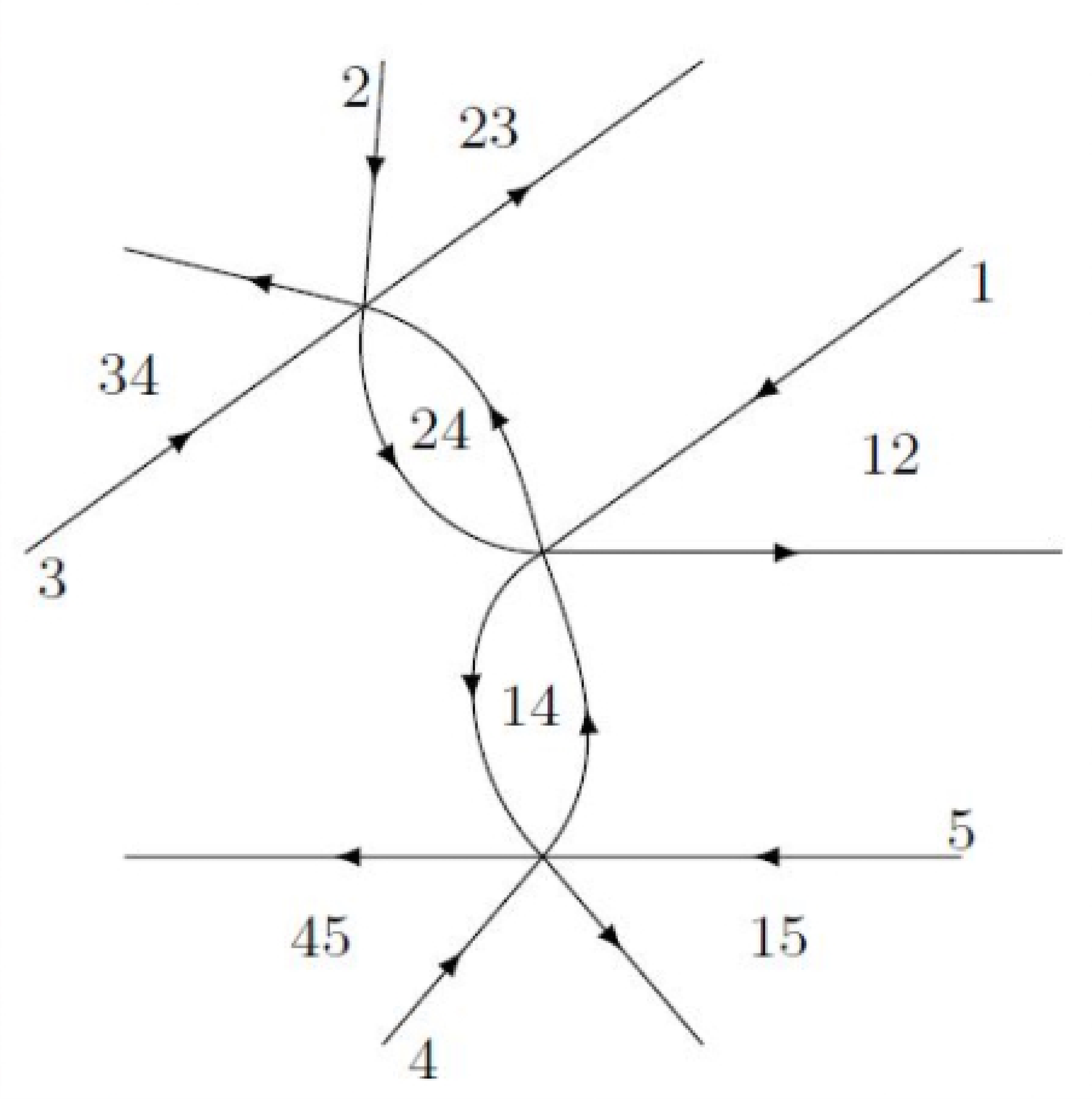
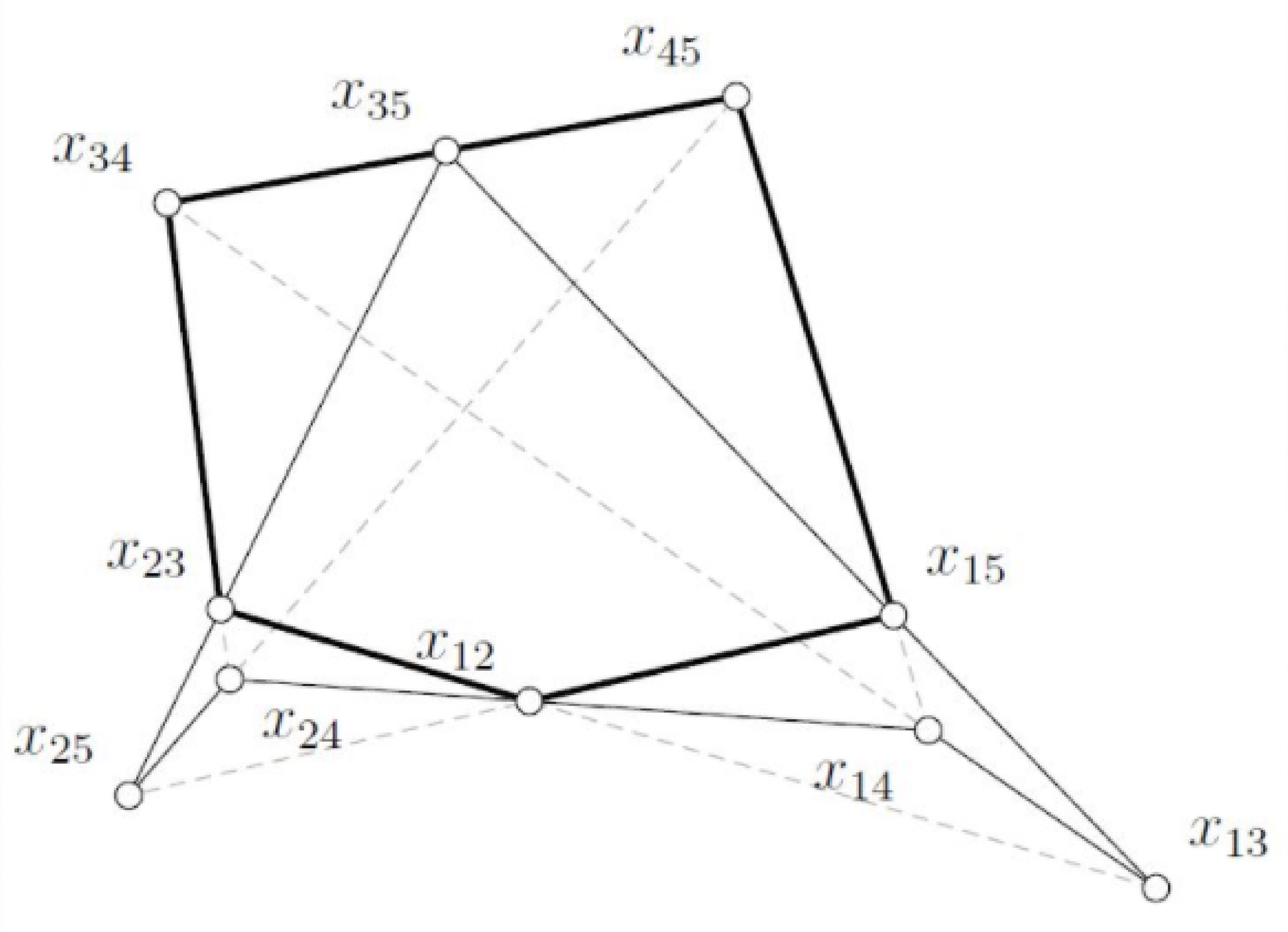
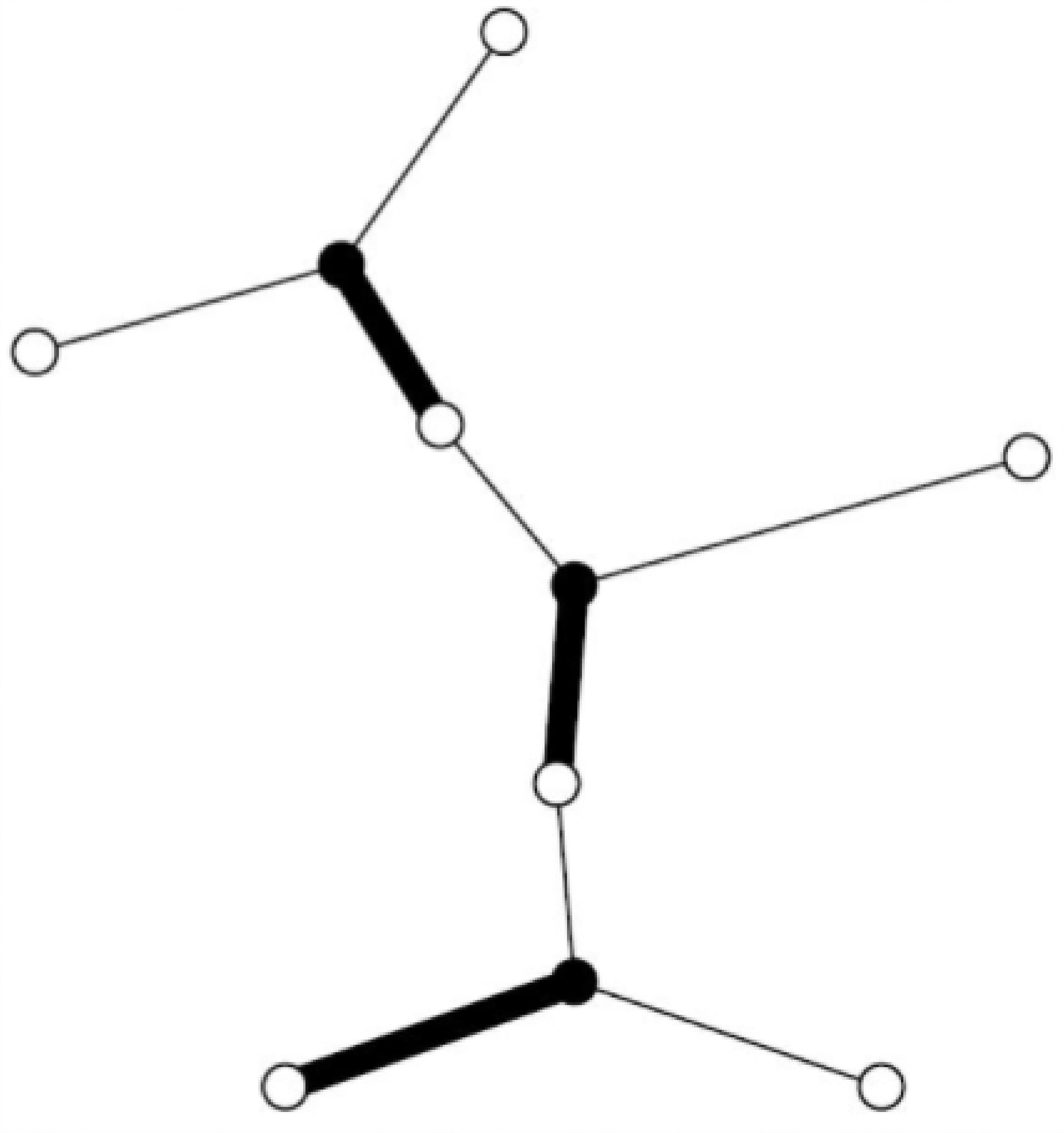
Triple Crossing Diagrams



Projective configurations

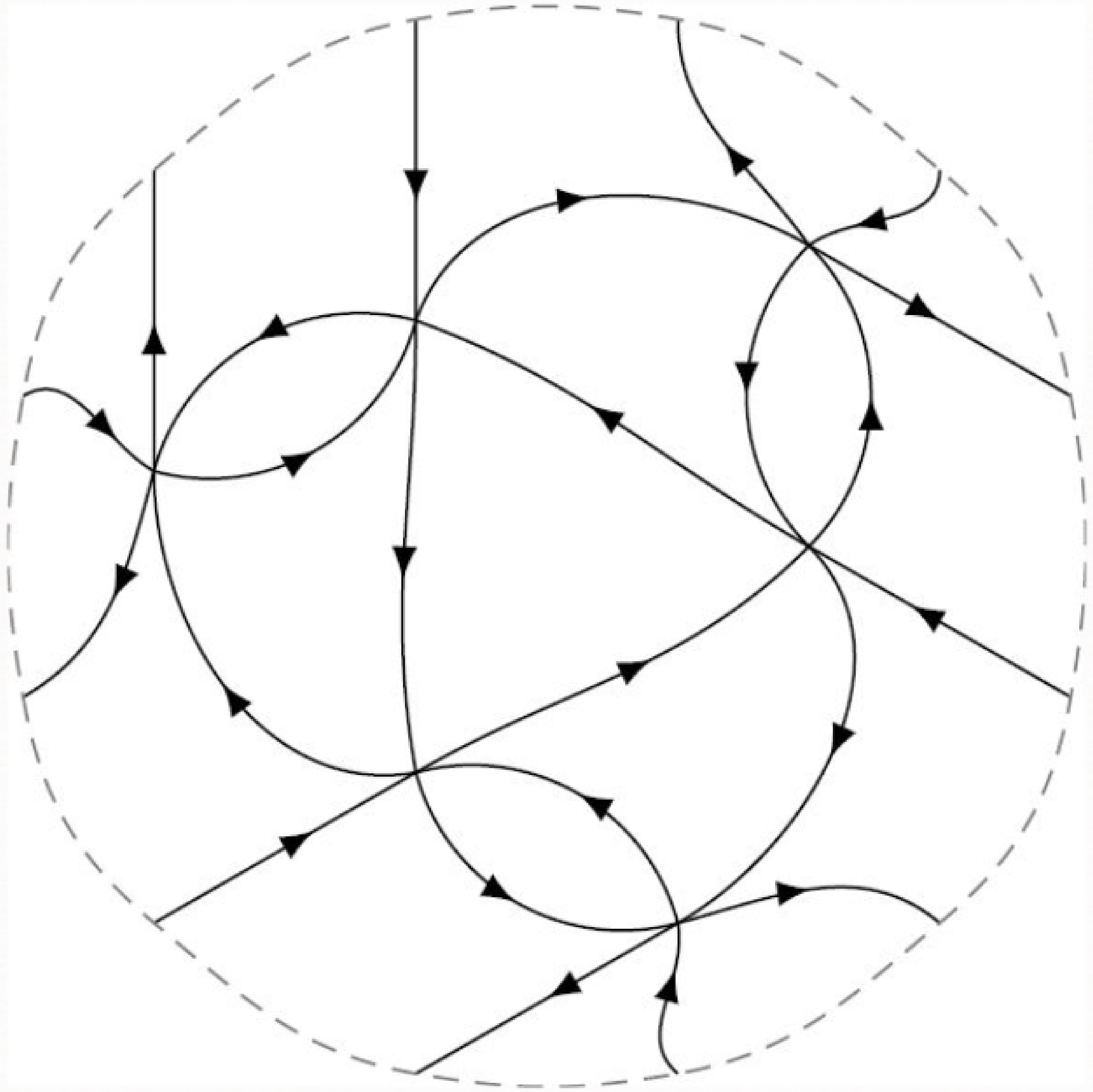


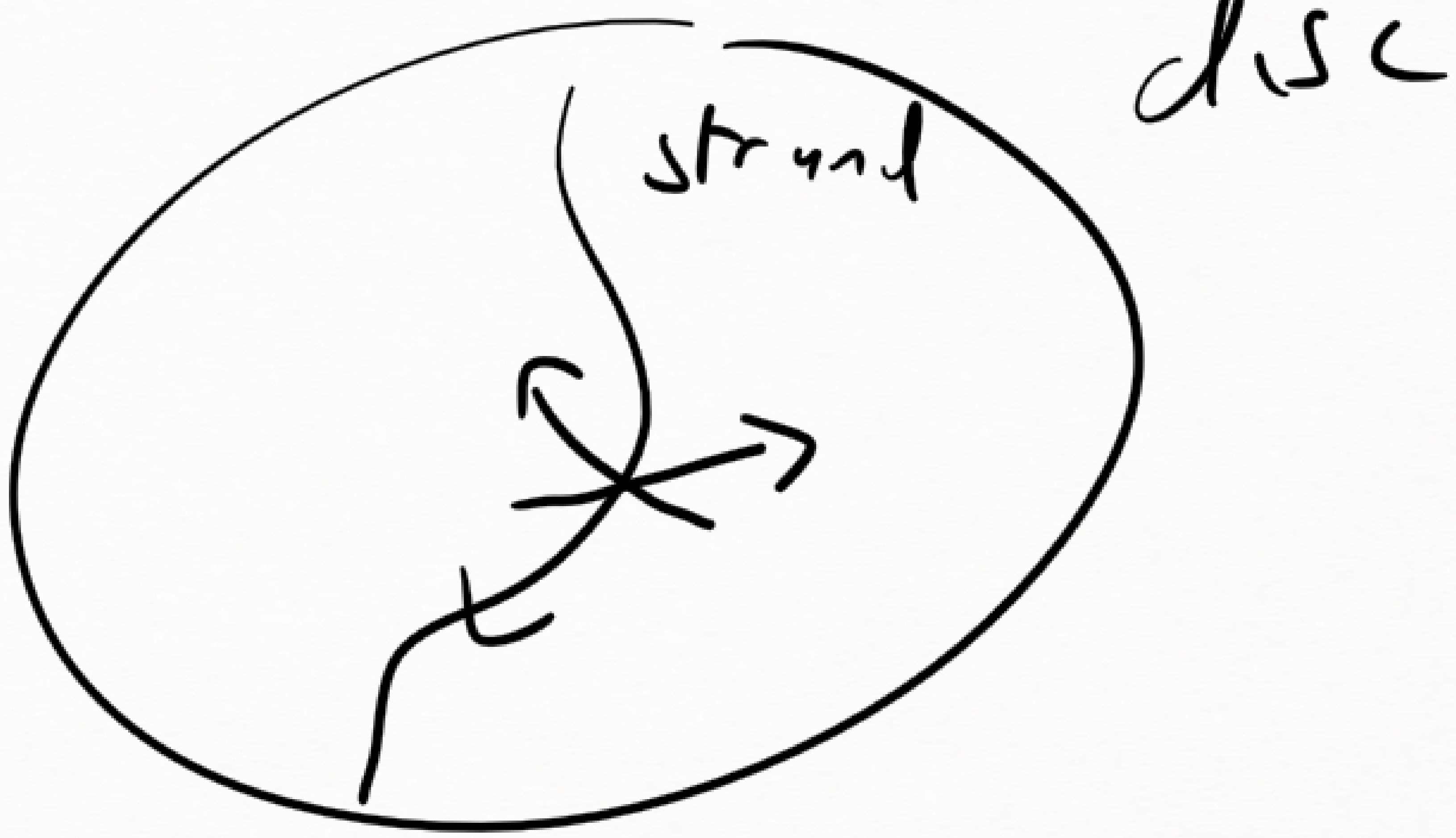
Dimers



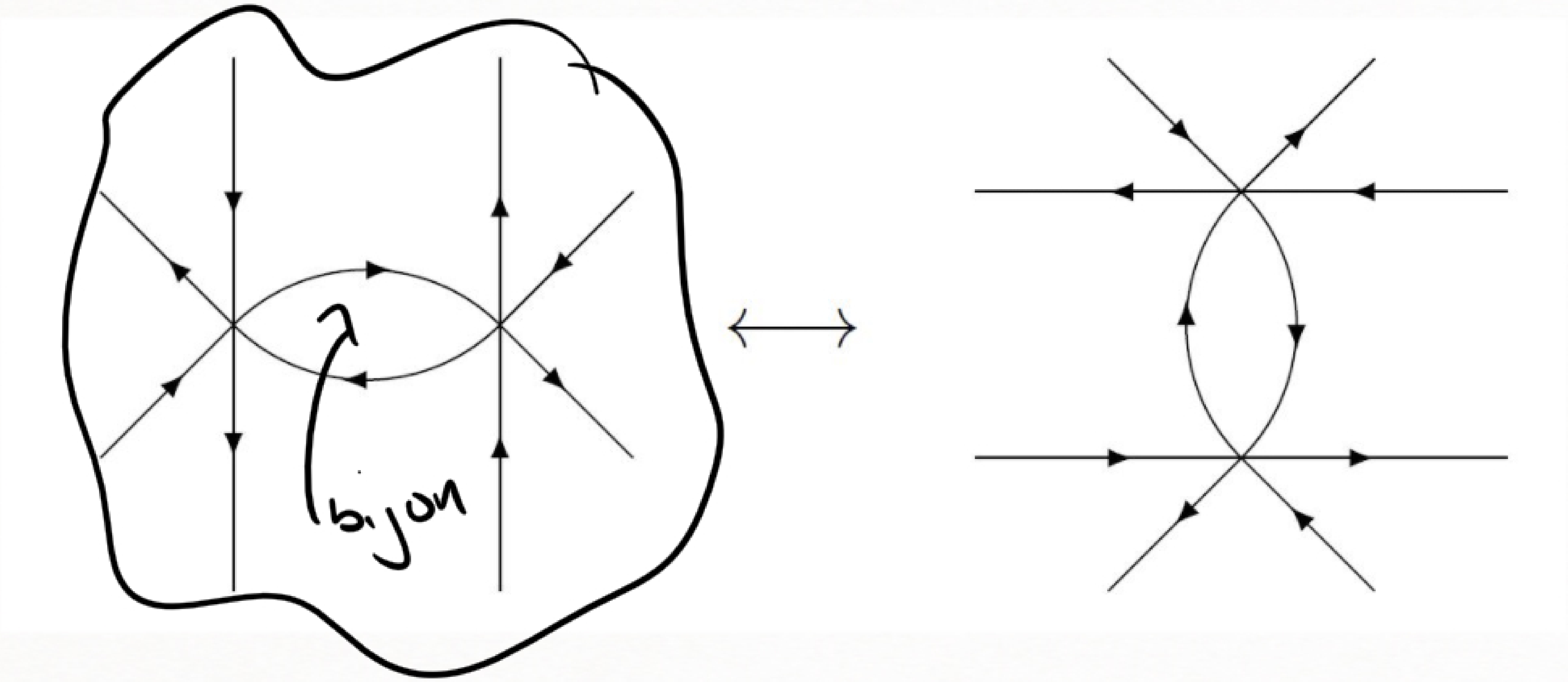
Joint work with Max Glick,
Pavlo Pylyavskyy, Sanjay Ramassamy

Triple crossing diagrams [Th]





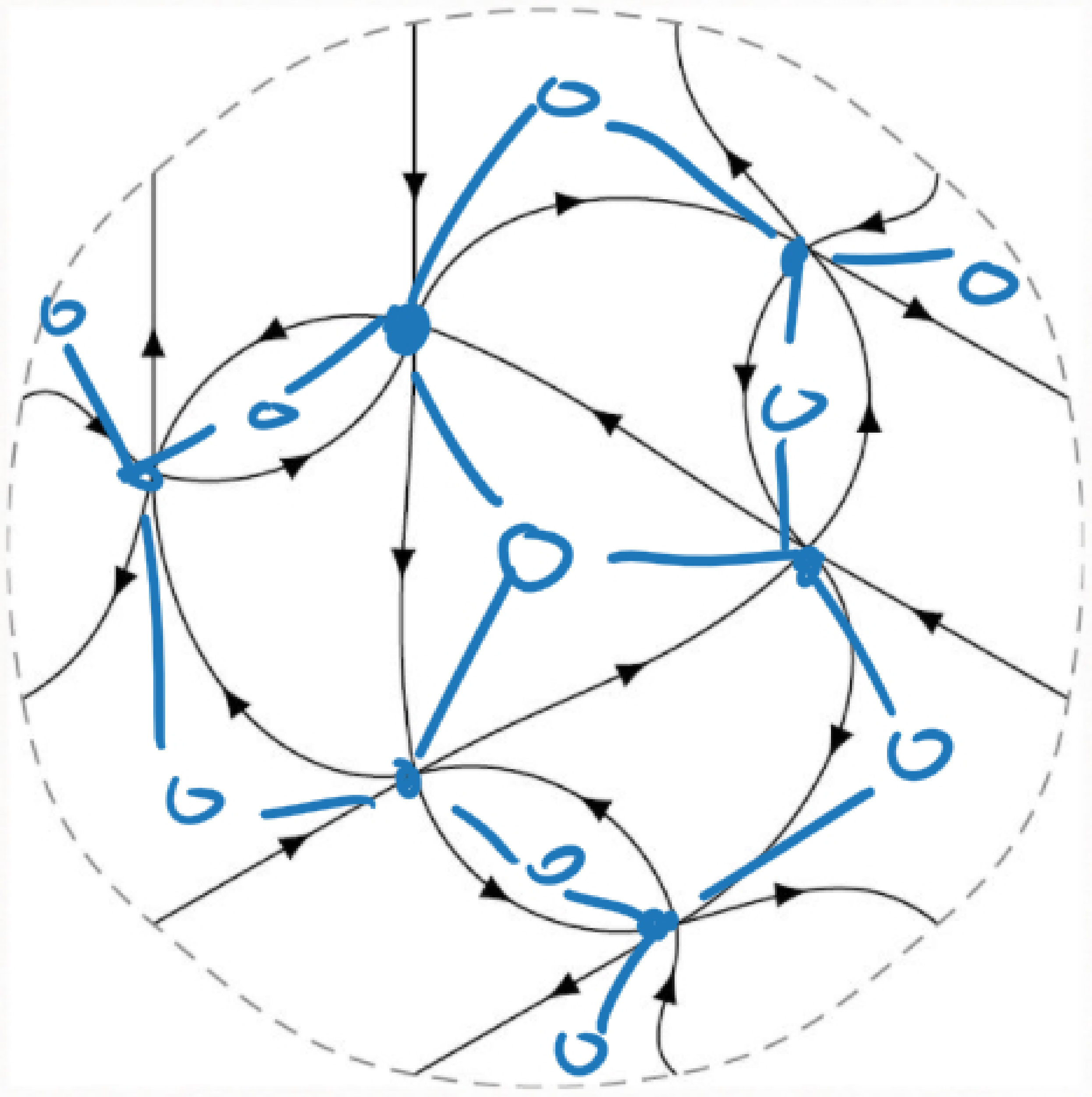
TCDs and 2-2 moves



Observations.

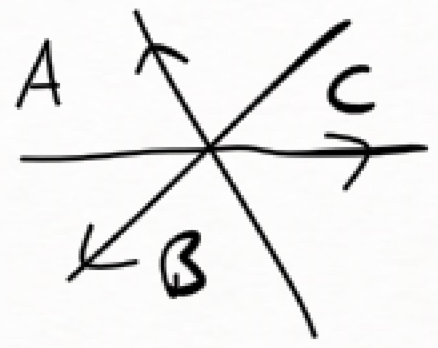
- preserves # intersections
- preserves orient bijon

TCD maps



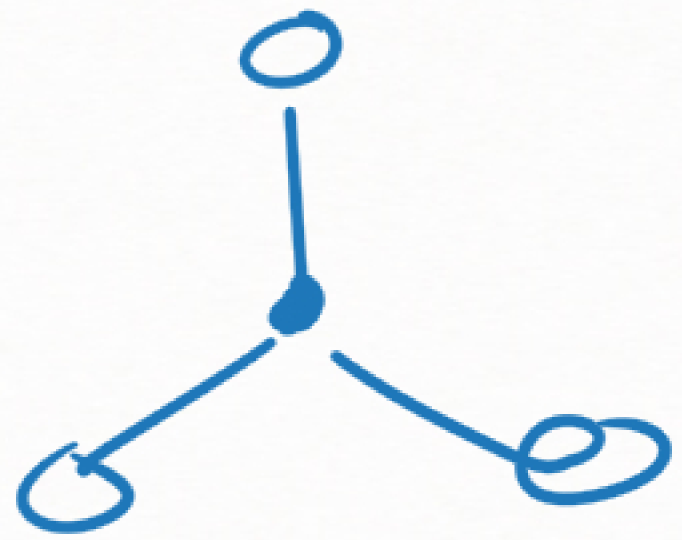
$\widetilde{\mathcal{T}}$

<u>Def</u>: Let $\mathcal{T}$ be a TCD and $\circlearrowleft$ its ccw faces.
Then $T : \circlearrowleft \to \mathbb{CP}^n$ is a **TCD map** if

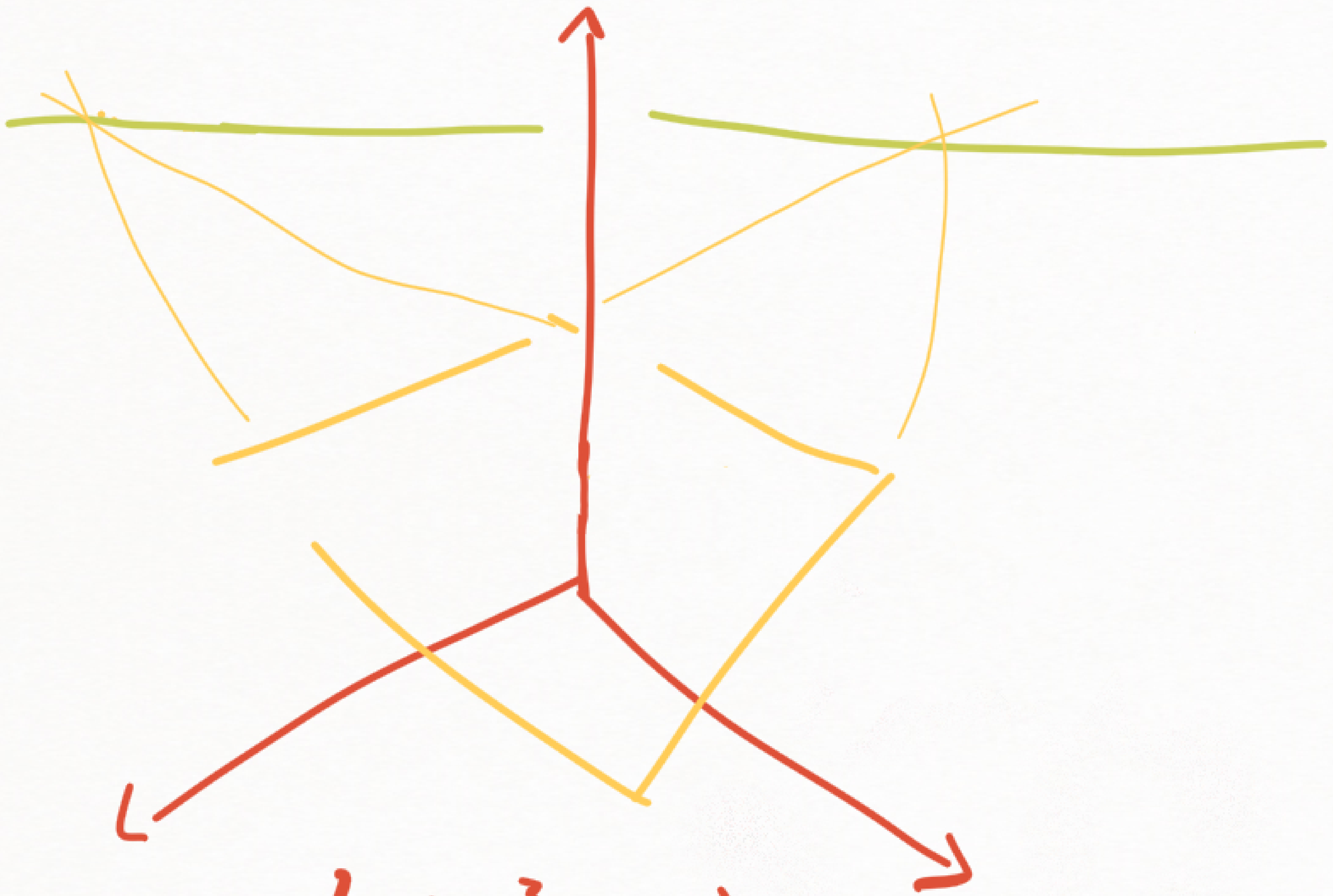


$T(A), T(B), T(C)$ are colinear at every intersection.

Fundamental unit:



Projective geometry



$\mathbb{C}P^2 = (\mathbb{C}^3 \setminus \{0\})/\sim$

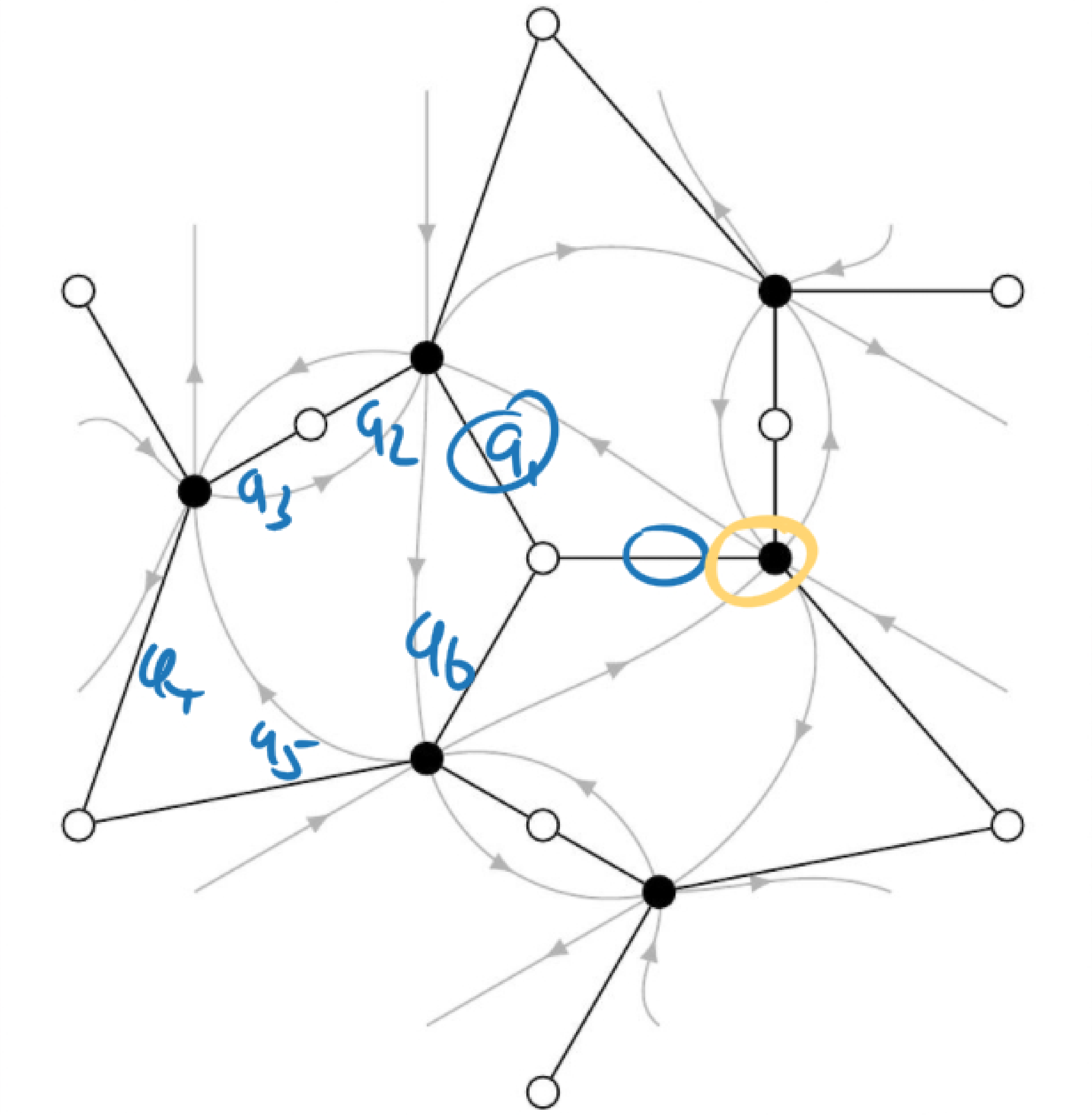
Def: $\mathbb{C}P^n = \{\mathbb{C}^{n+1} \setminus \{0\}\}/\sim$

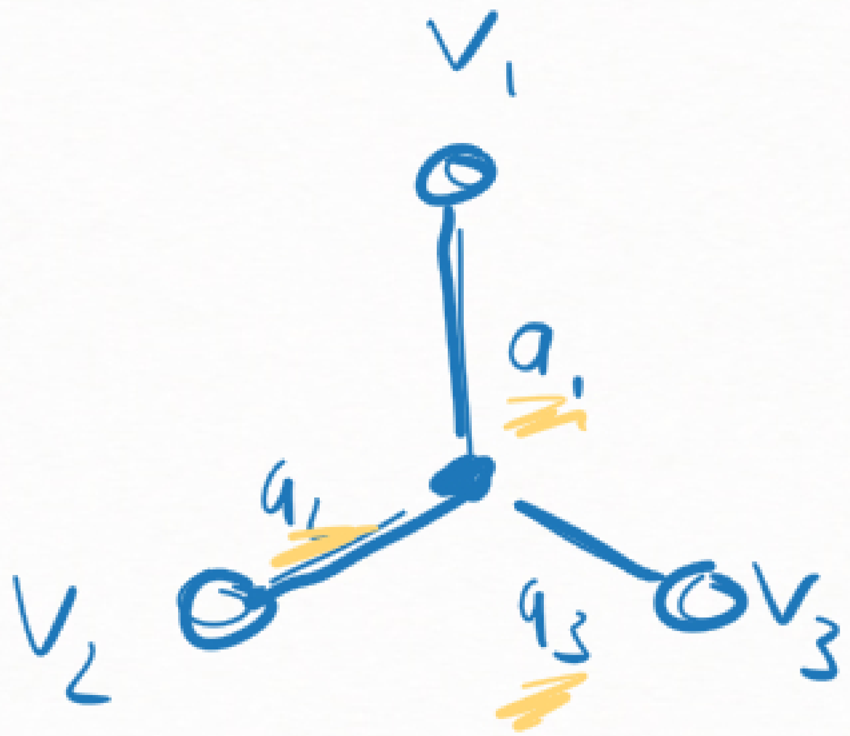
$a, b \in \mathbb{C}^{n+1}$: $a \sim b \iff \exists \lambda \in \mathbb{C}: a = \lambda b$

a are homogeneous coordinates of point $[a]$

scaling ~ gauge freedom

Linear relations





$\Rightarrow \exists\ a_1, a_2, a_3 \in \mathbb{C}$:

$$a_1 V_1 + a_2 V_2 + a_3 V_3 = 0$$

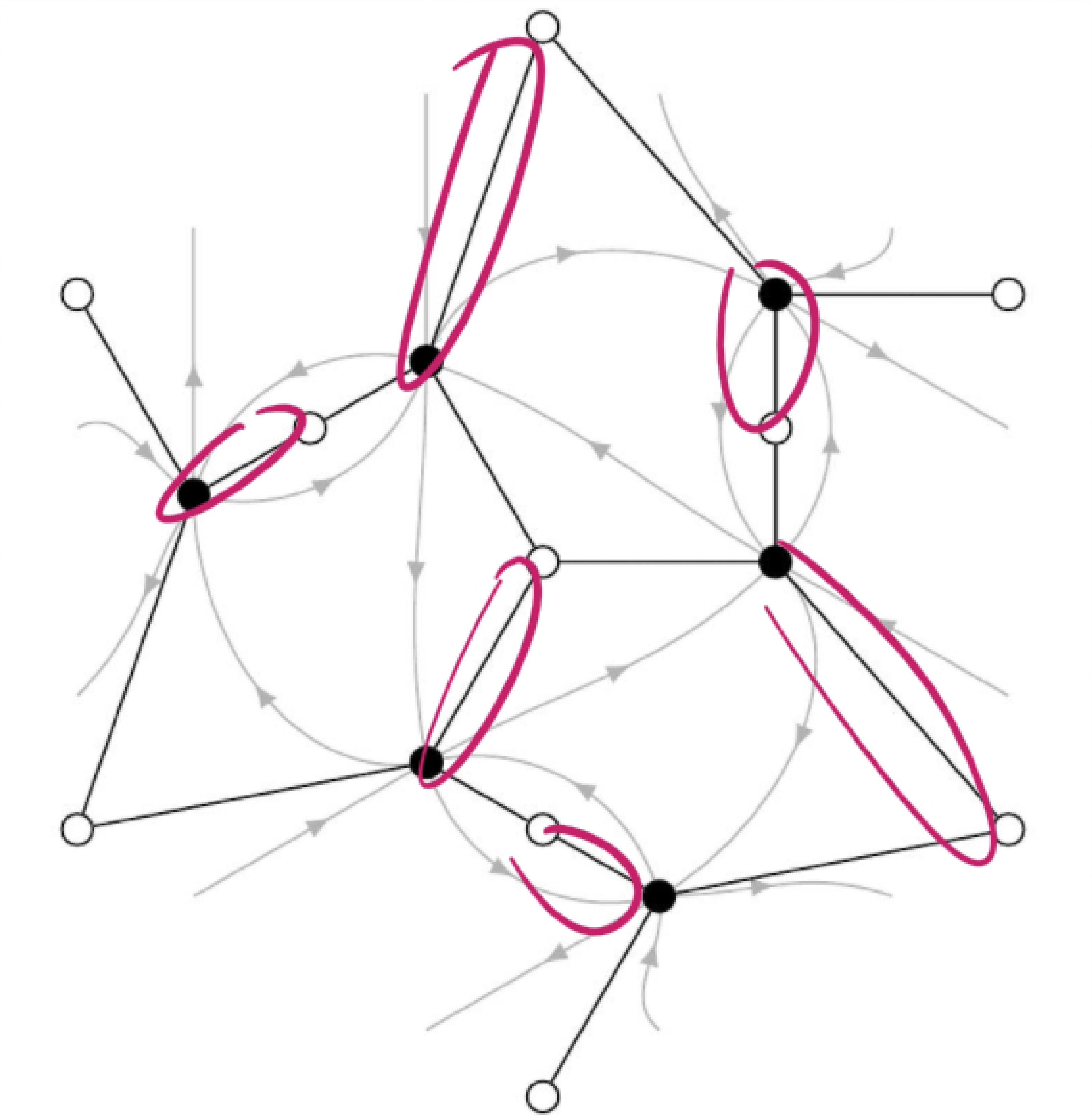
Rearrange $\Rightarrow$ $\mu\left[a_1 \lambda_1^{-1}(\lambda_1 v_1) + a_2 \lambda_2^{-1}(\lambda_2 v_2) + a_3 \lambda_3^{-1}(\lambda_3 v_3)\right] = 0$

$\forall \mu, \lambda_v \in \mathbb{C}$

$$\frac{a_1}{a_2} \quad \frac{a_3}{a_4} \quad \frac{a_5}{a_6}$$

"altern. products"

Dimers



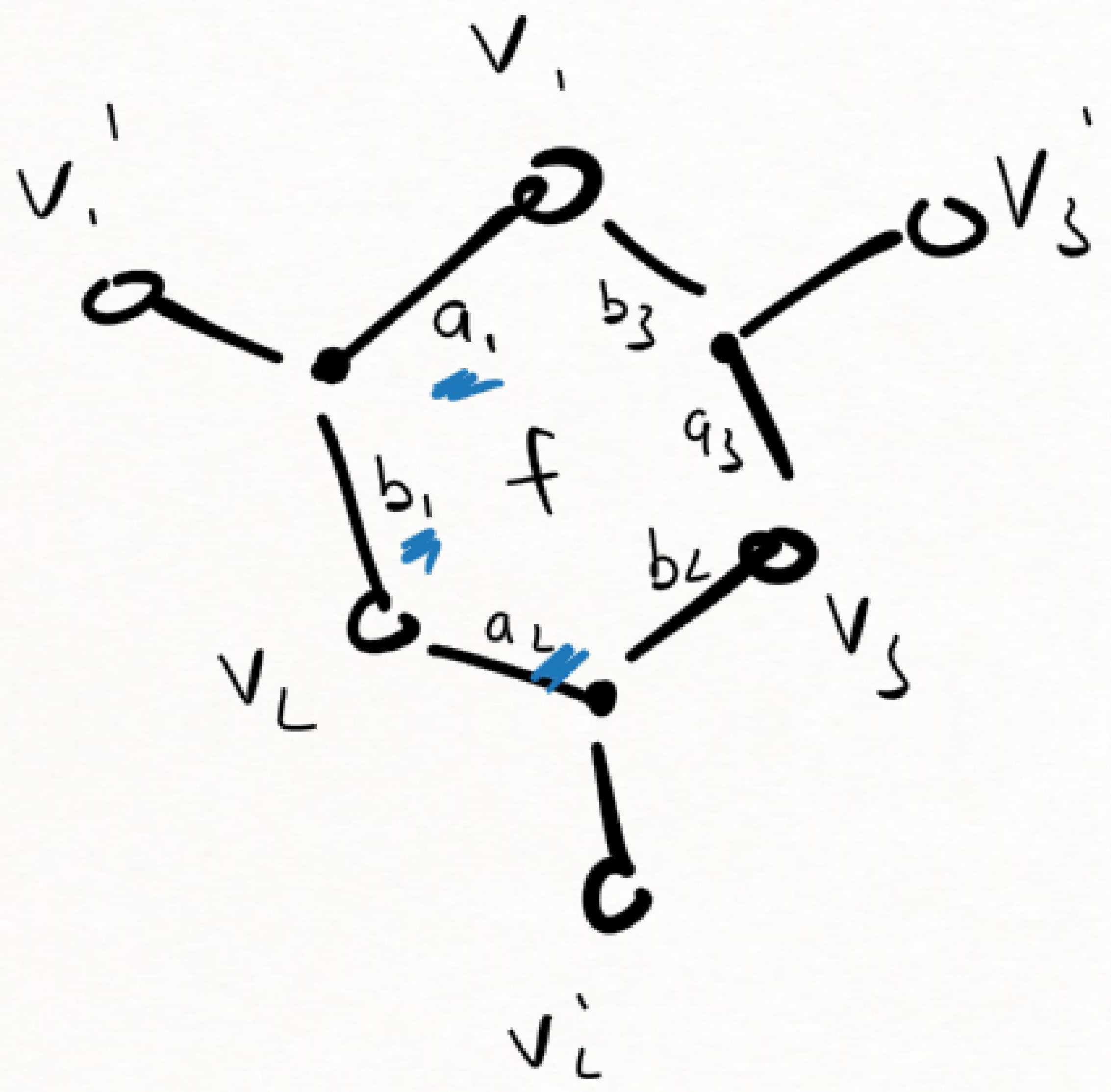
Def: Almost perfect matching

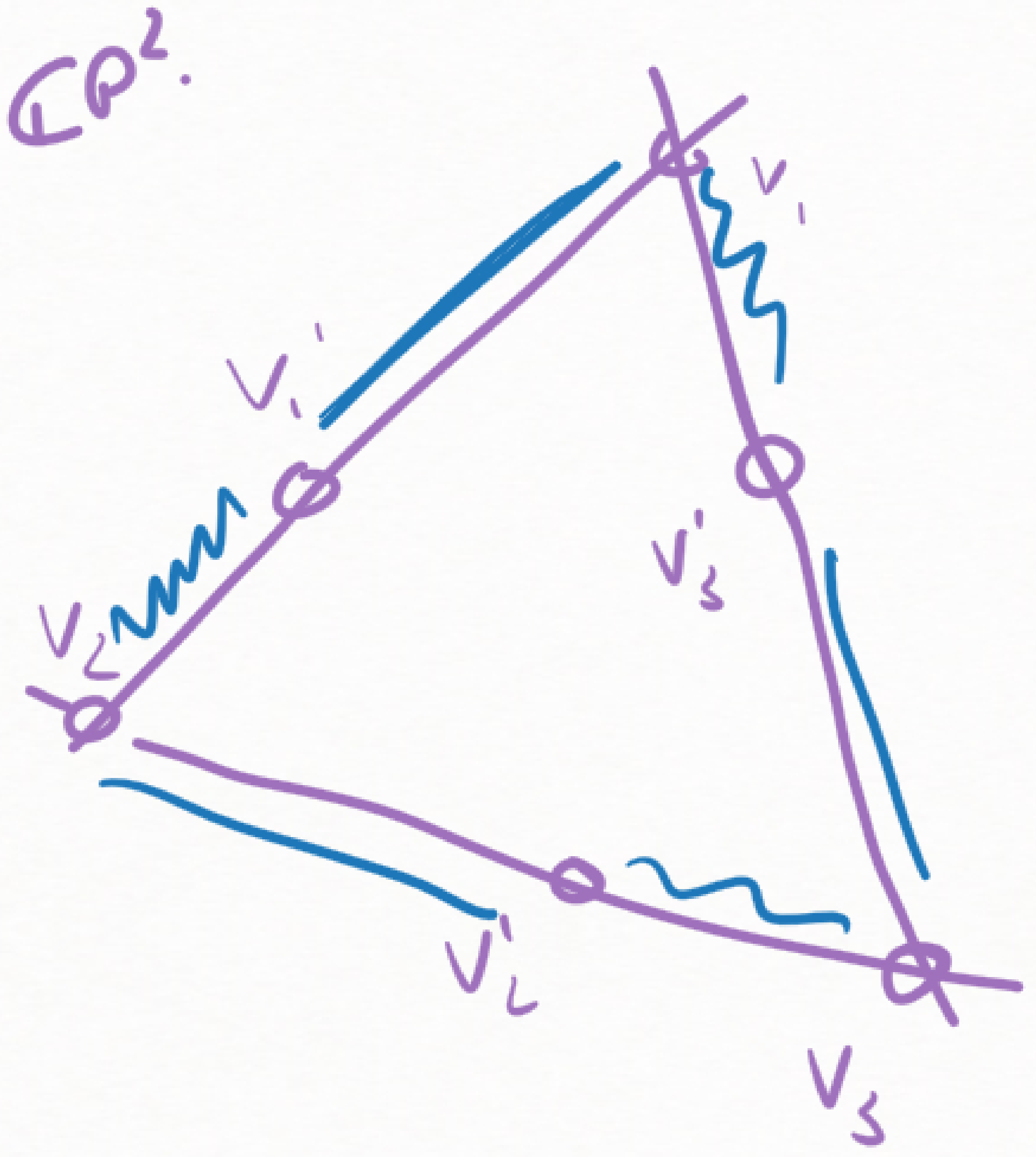
- All interior vertices matched
- All but #W - #B boundary vertices matched

$\longleftarrow$ Gauge behaviour

Moves $\longrightarrow$

Multi-ratios





Def $$\boxed{X_f = (-1)^{n+1} \frac{b_1}{a_1} \frac{b_2}{a_2} \cdots \frac{b_n}{a_n}}$$

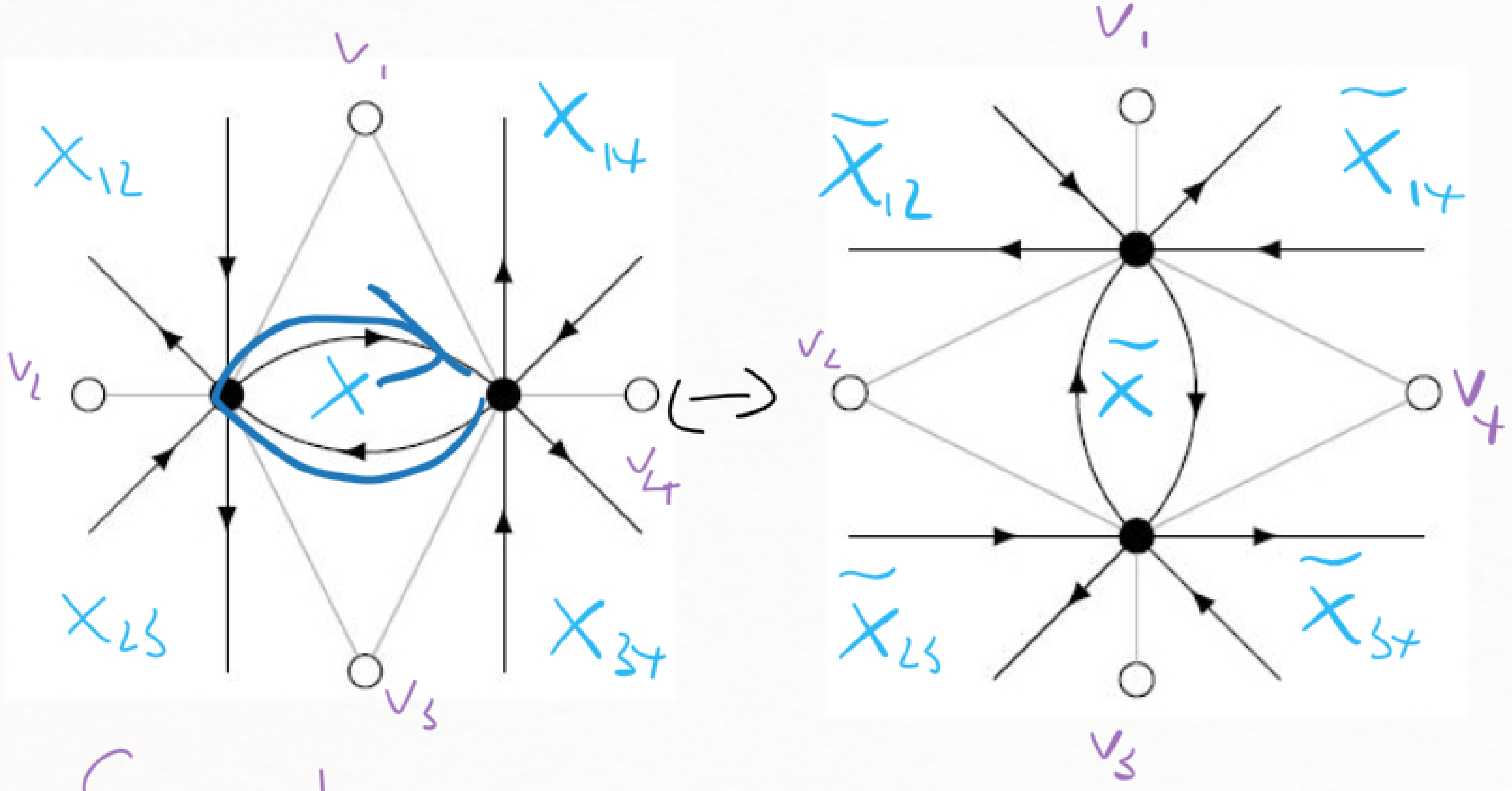
Def $$\boxed{mr(v_1, v_1', v_2, v_2', \ldots, v_n, v_n')} = \frac{v_1 - v_1'}{v_1' - v_2} \frac{v_2 - v_2'}{v_2' - v_3} \cdots \frac{v_n - v_n'}{v_n' - v_1}$$

Lemma $X_f = (-1)^{n+1} mr(v_1, v_1', \ldots, v_n, v_n')$

Lemma Multi-ratios are invariant under projective transformations

Spider-move ↻

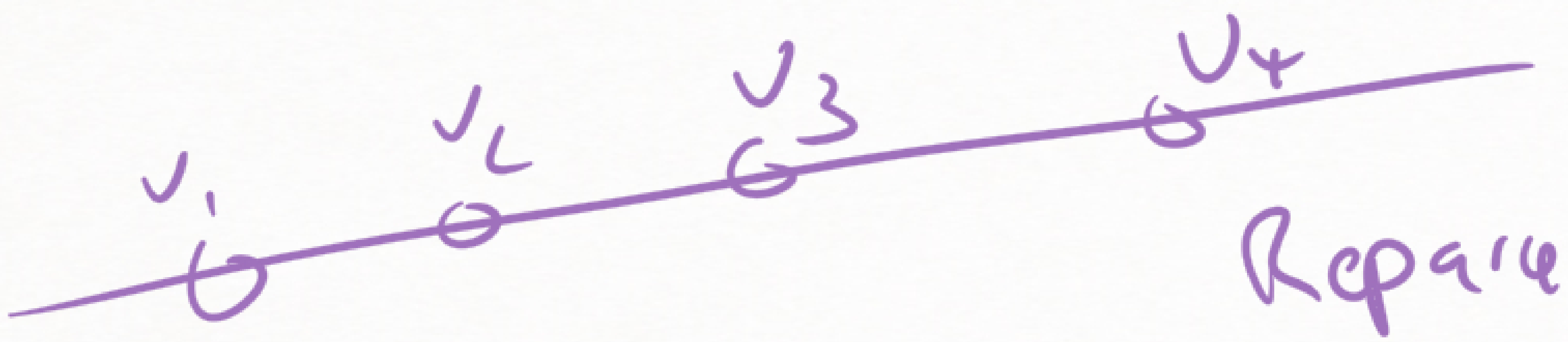
/ CW 2-2 move



$$\left.\begin{aligned}
\tilde X &= X^{-1} \\
\tilde X_{12} &= X_{12}(1+X^{-1})^{-1} \\
\tilde X_{23} &= X_{23}(1+X) \\
\tilde X_{34} &= X_{34}(1+X^{-1})^{-1} \\
\tilde X_{14} &= X_{14}(1+X)
\end{aligned}\right\}$$

Rules for Z-invariant move of the dimer model.

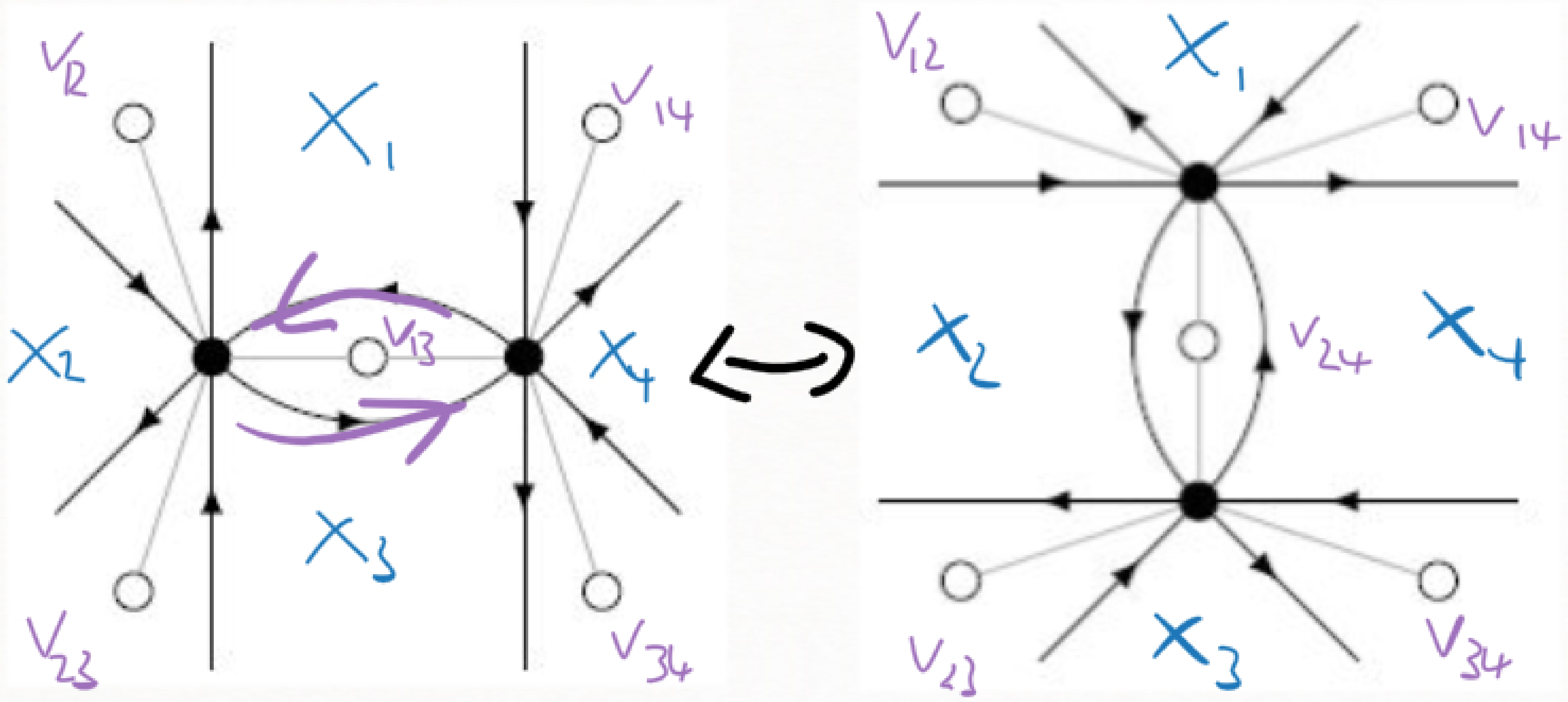
Geometry:



Reparametrization

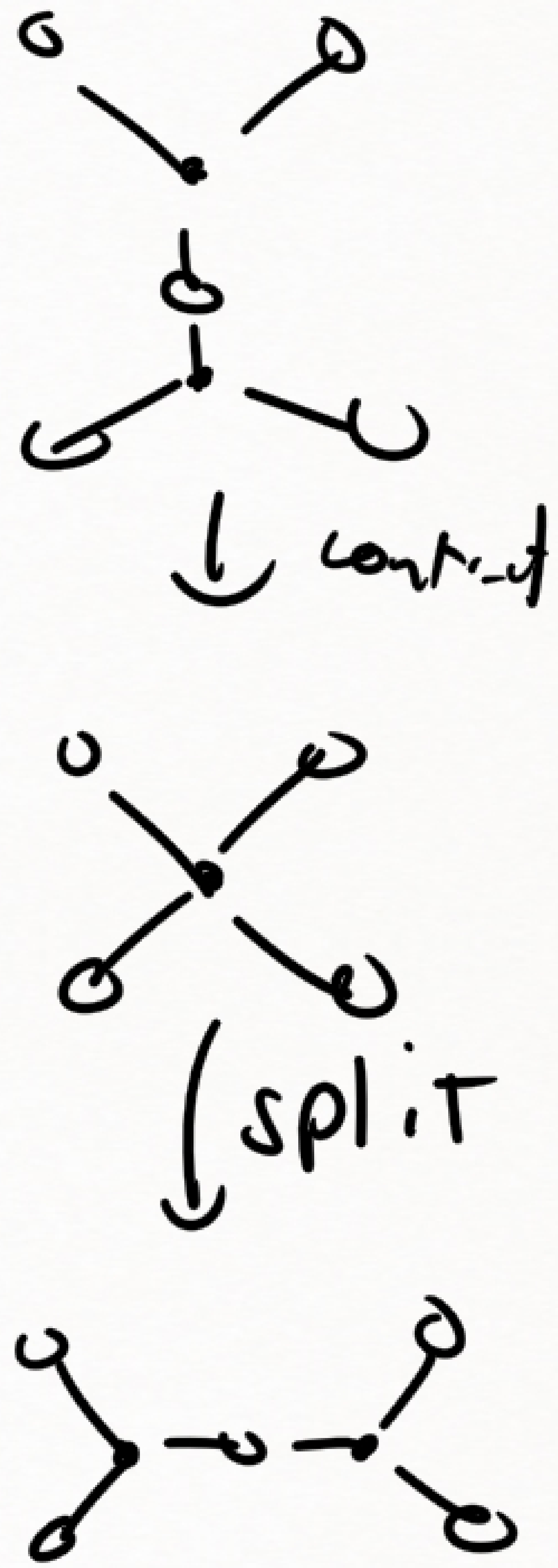
Resplit ↺

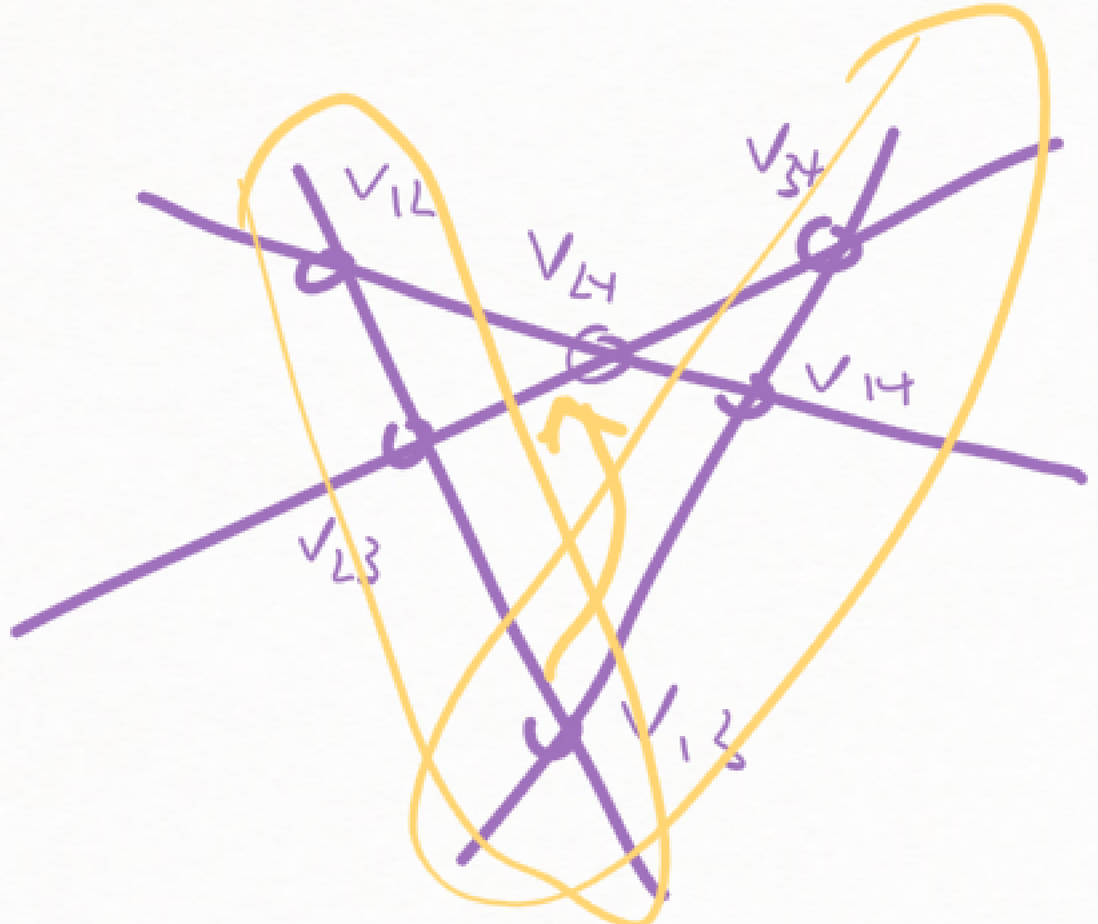
/2-2 move ccw



X - variables are invariant

Geometry changes

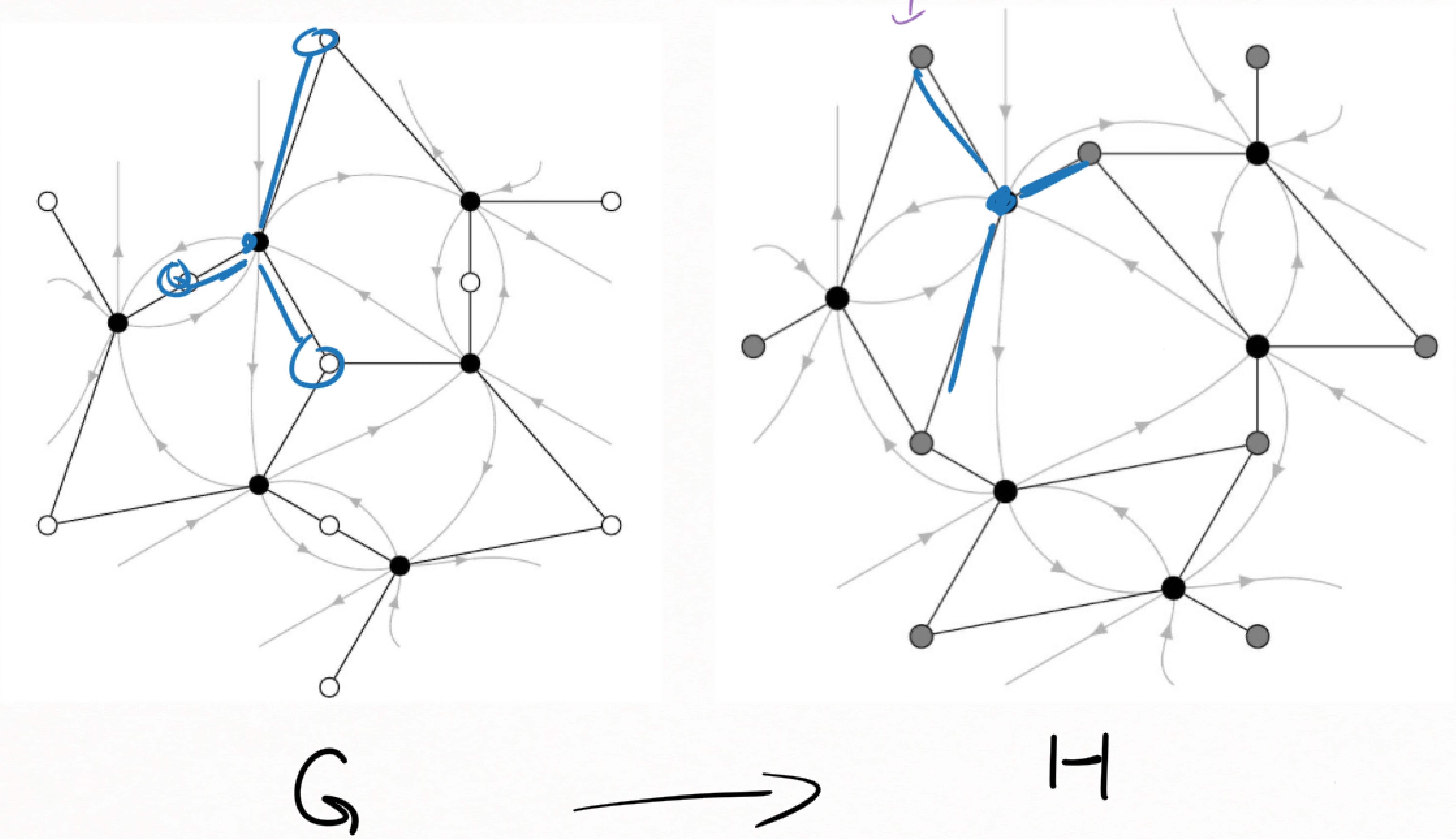




Menelaus configuration or complete quadrilateral

Affine Dimers

not a point in $\mathbb{C}P^n$

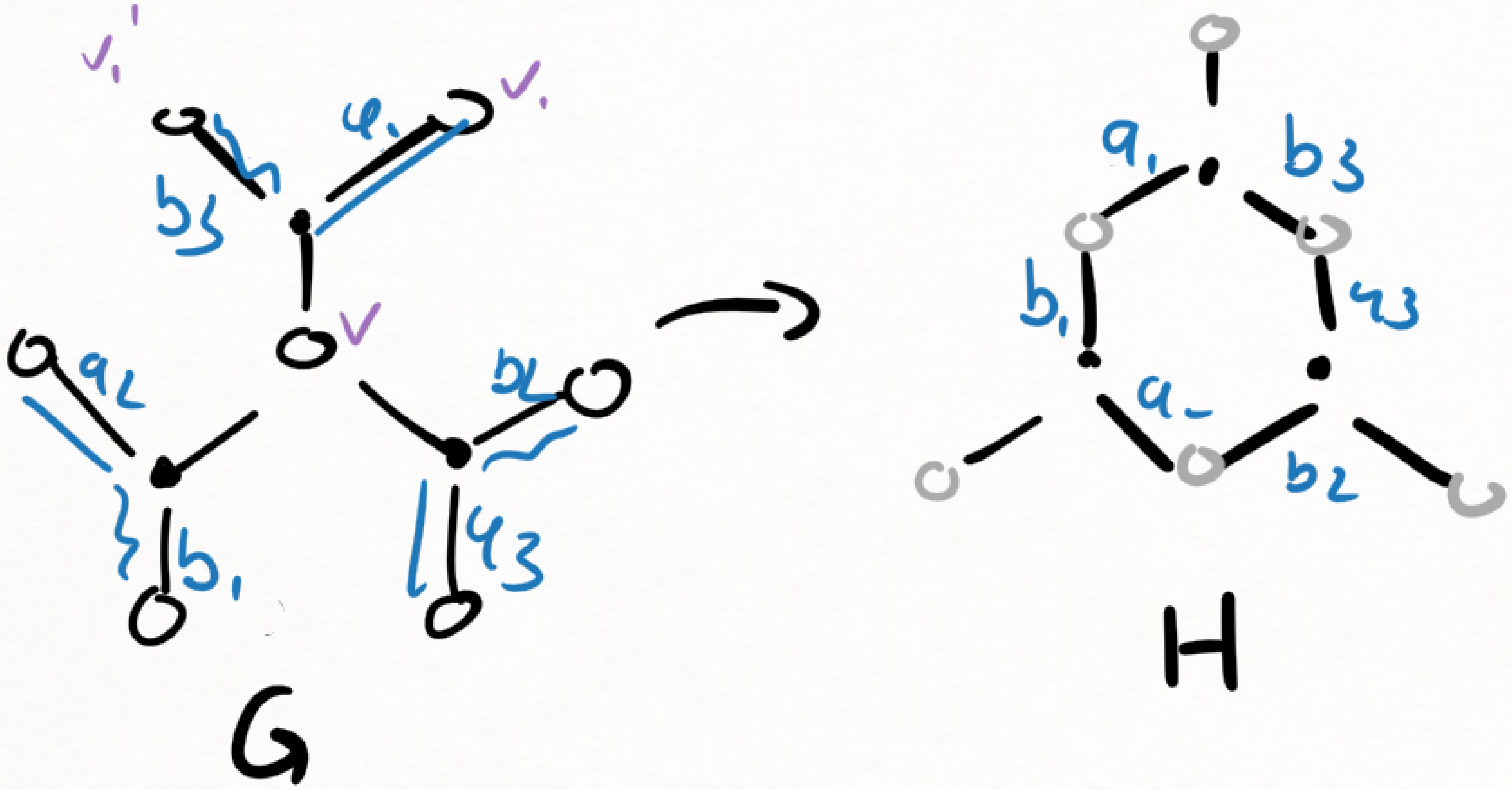


Affine chart

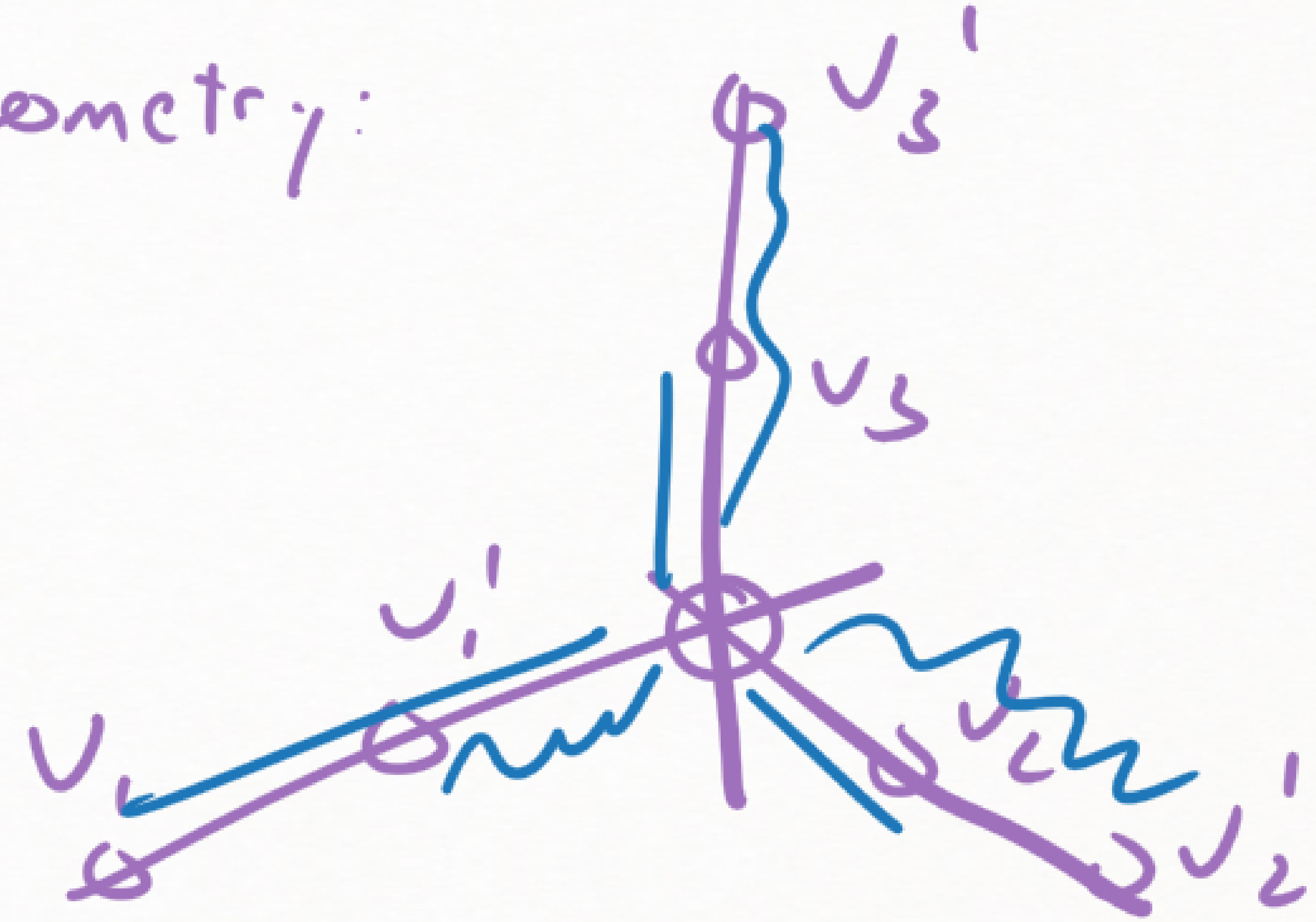
$$a_1 \begin{pmatrix} v_{11} \\ v_{12} \\ \vdots \\ \boxed{1} \end{pmatrix} + a_2 \begin{pmatrix} v_{21} \\ v_{22} \\ \vdots \\ \boxed{1} \end{pmatrix} + a_3 \begin{pmatrix} \vdots \\ \boxed{1} \end{pmatrix} = 0$$

$$\Rightarrow \underline{a_1 + a_2 + a_3} = 0$$

Affine variables Y



Geometry:

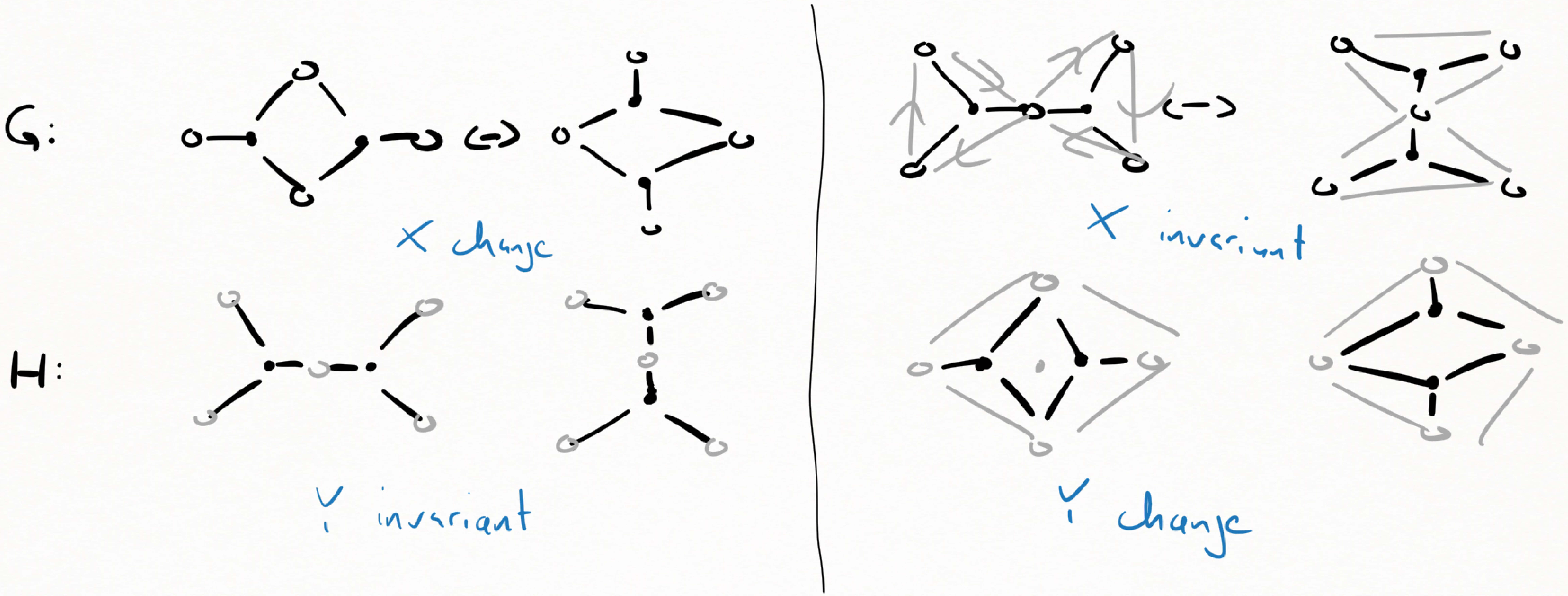


Alternating length ratio

'star-ratio' Y_v

It is affine invariant

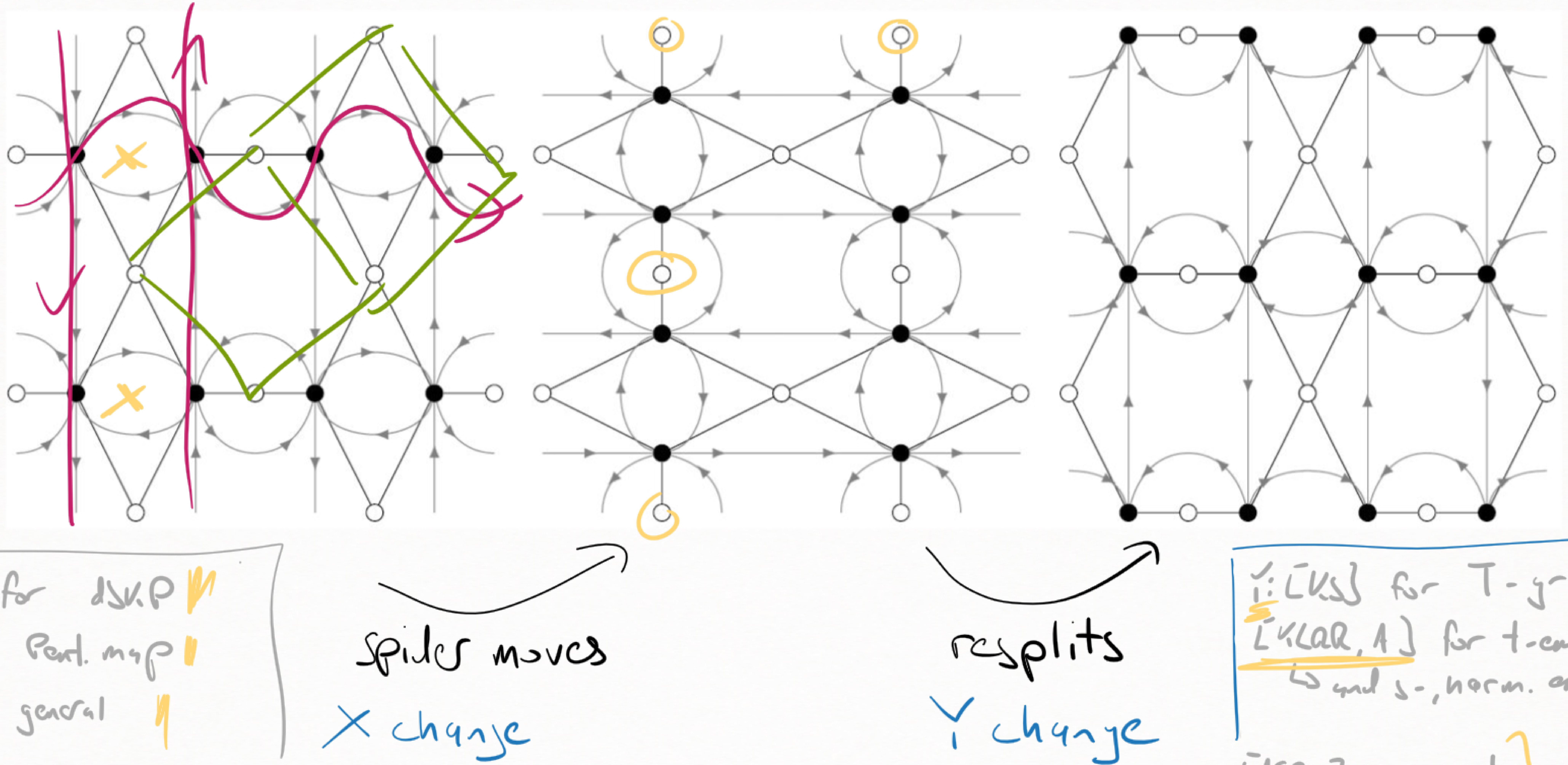
Affine + Projective 2-2 moves



Geometry invariant

Geometry changes

Example: $\boxed{\mathbb{Z}^2}$ combinatorics

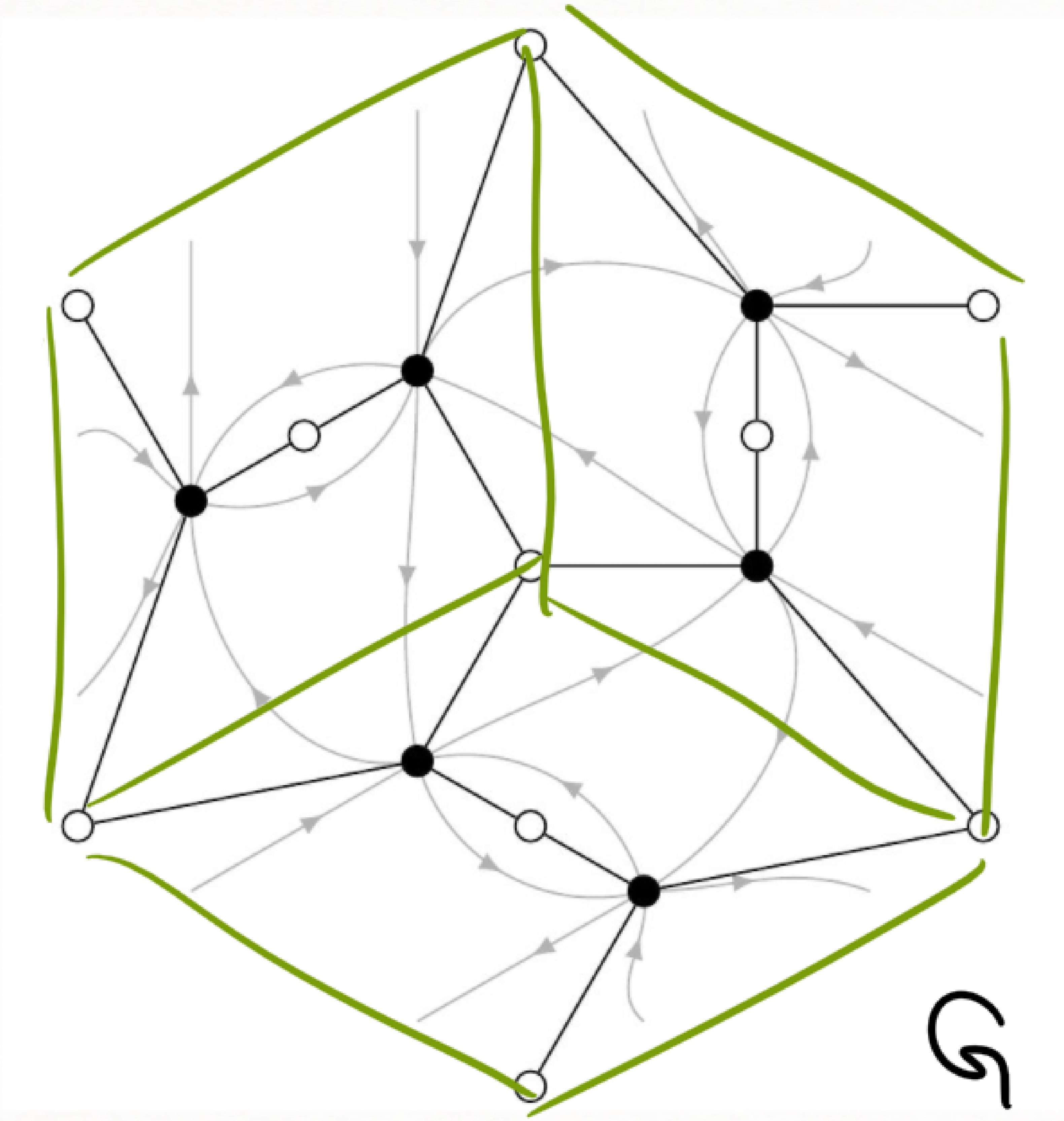


X: [ABS] for dSKP
[G] for Pent. map
[AGPR] general

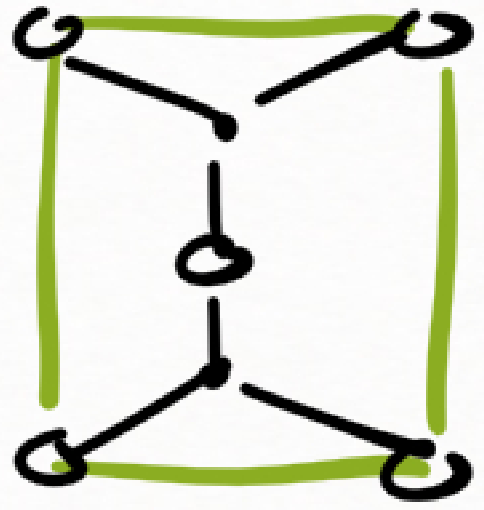
Y: [KS] for T-graphs
[KLRR, A] for t-emb.
↳ and s-, norm. emb

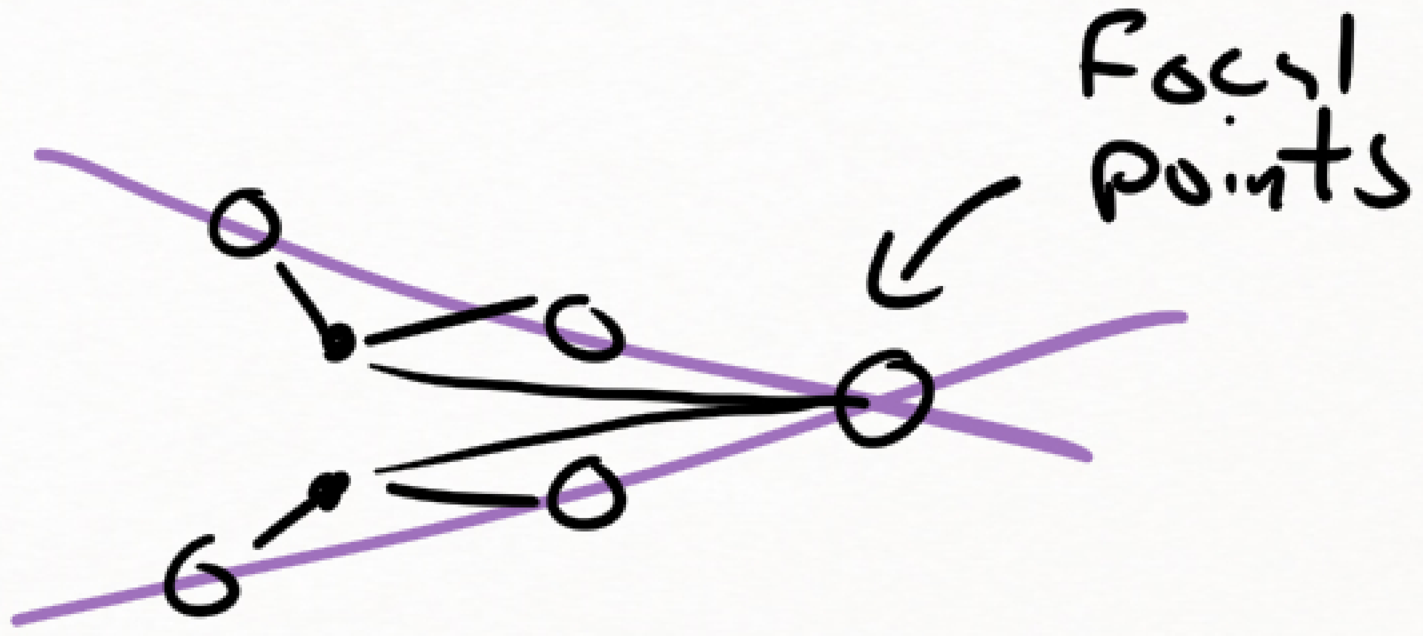
[AGR?] general

Example Q-nets



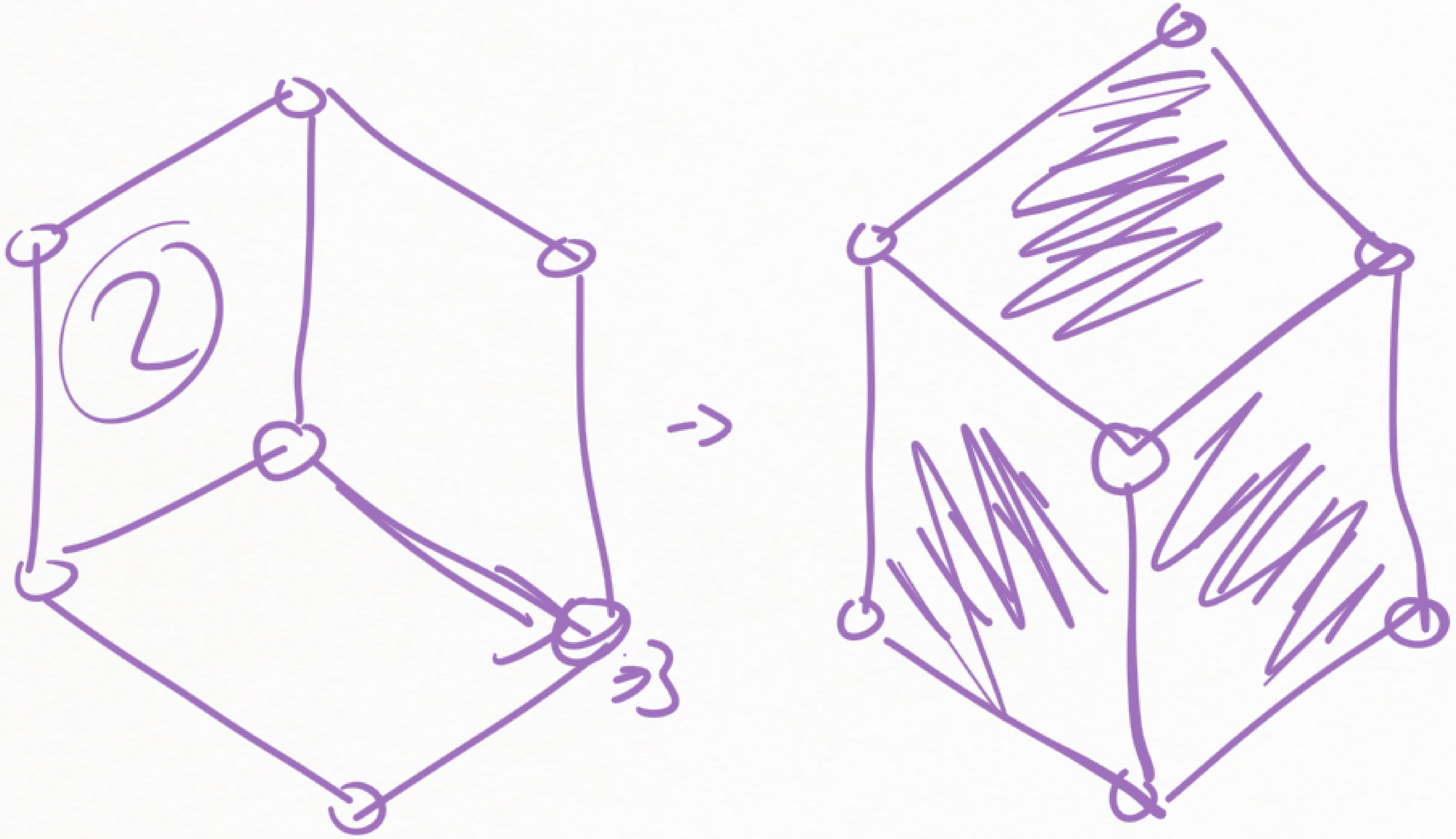
Def A map from a quadgraph to $\mathbb{CP}^n$ such that every quad is planar $(n>2)$



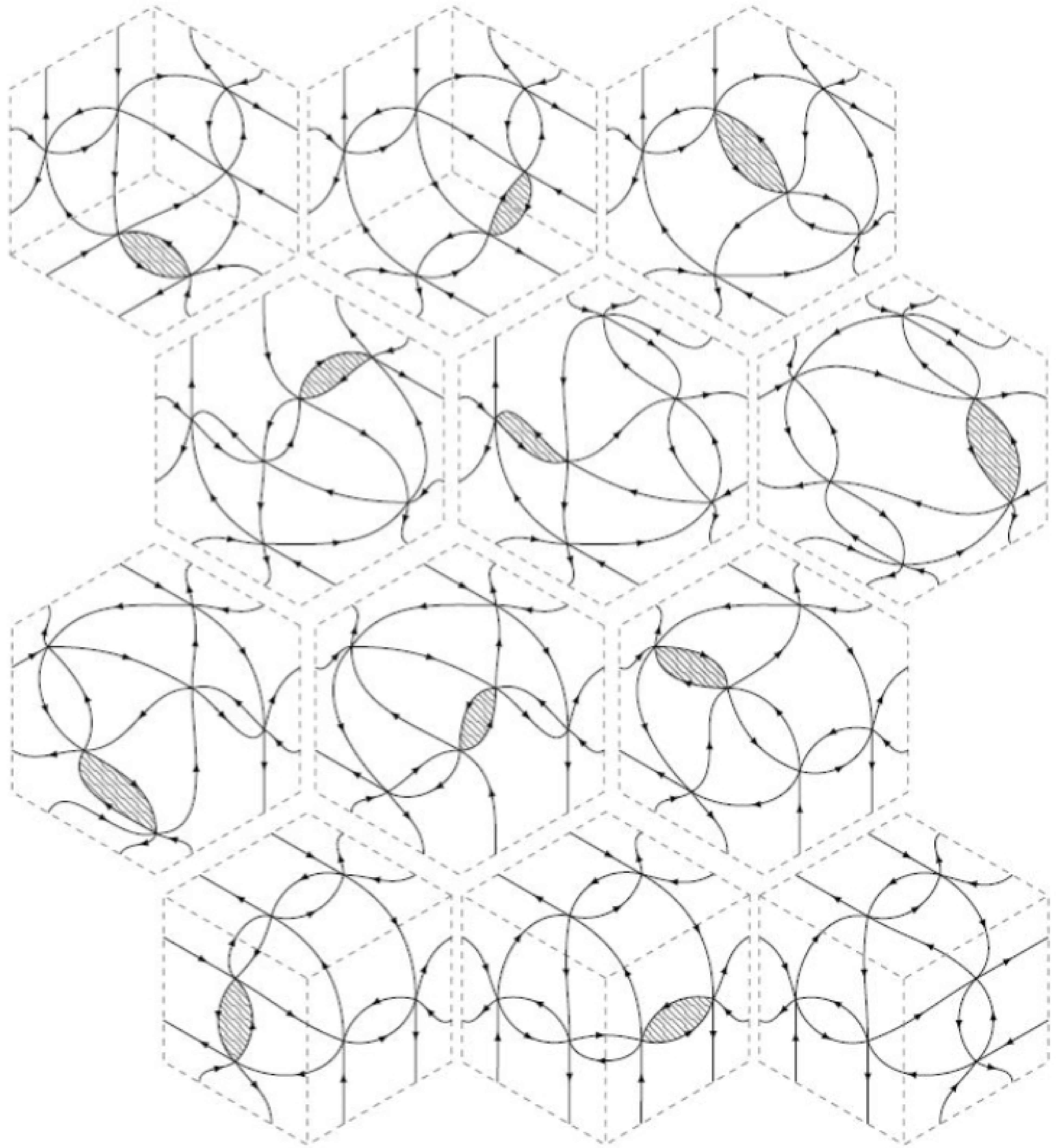


is a Q-net

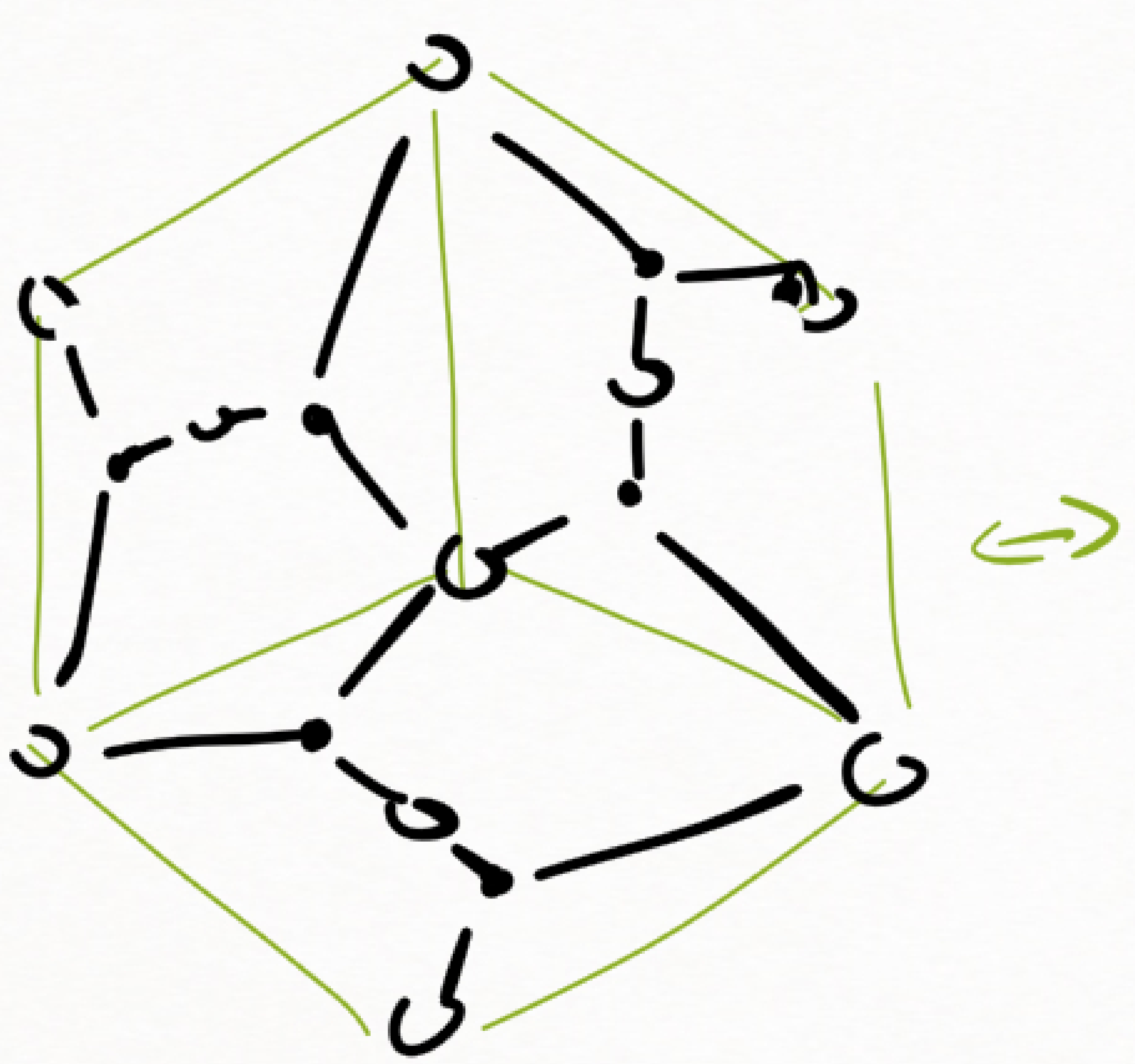
Q-net cubeflip



2-2 move sequence:

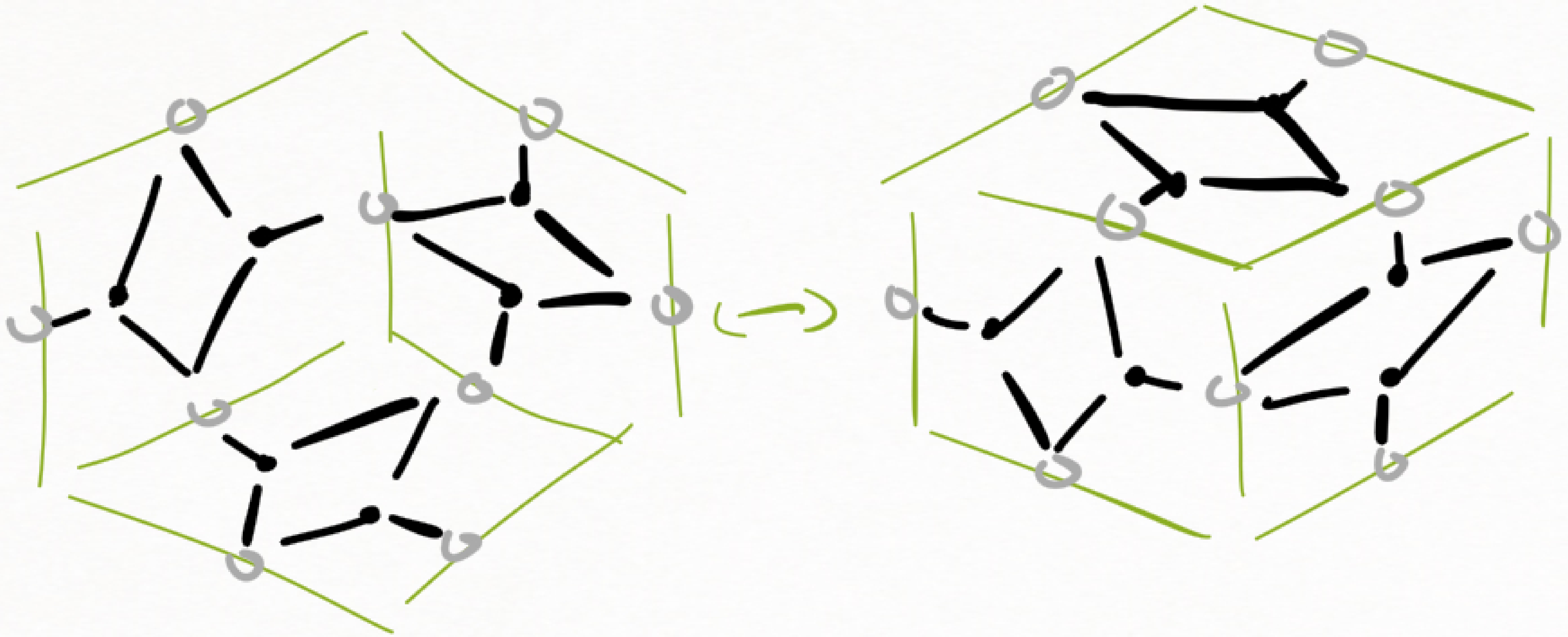


G:

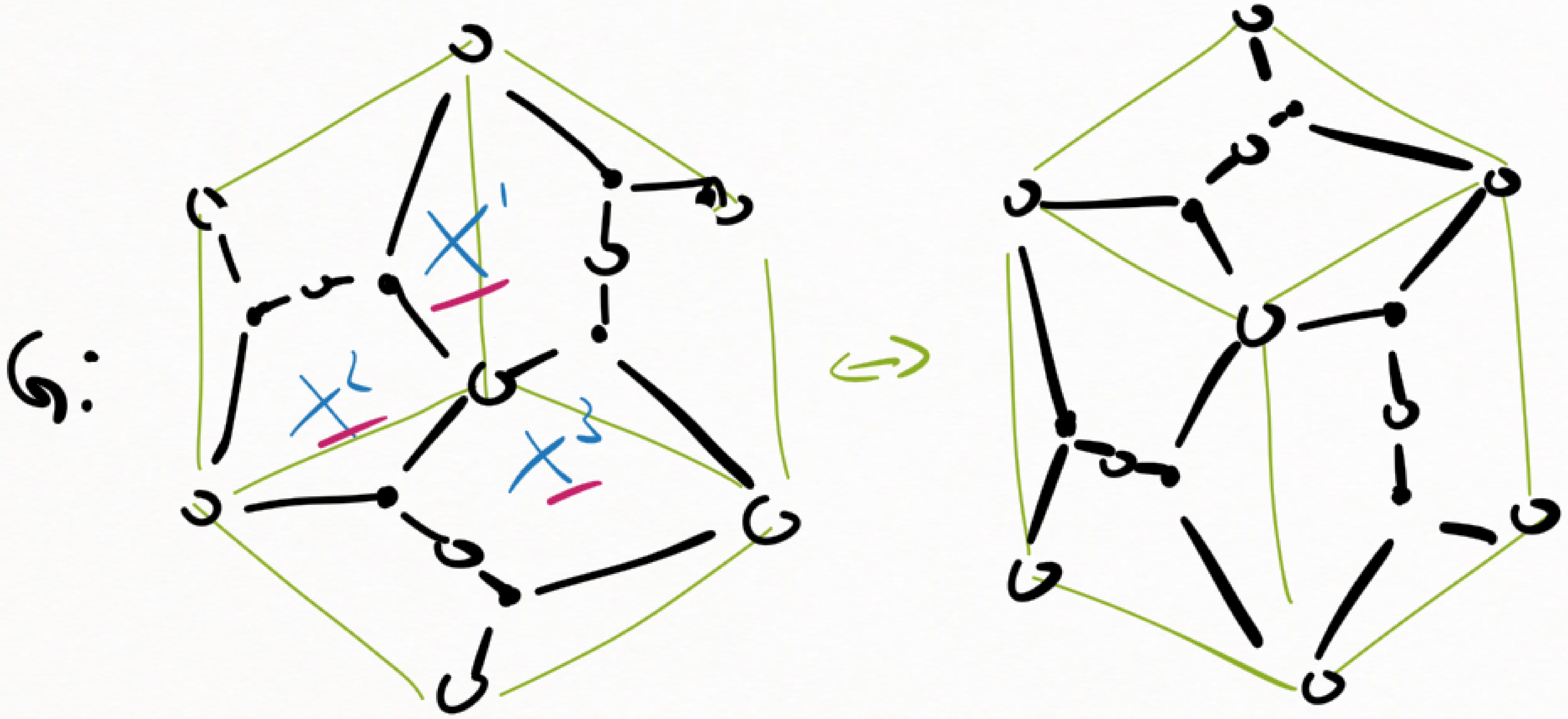


← Combinatorics that admit spanning trees

H:



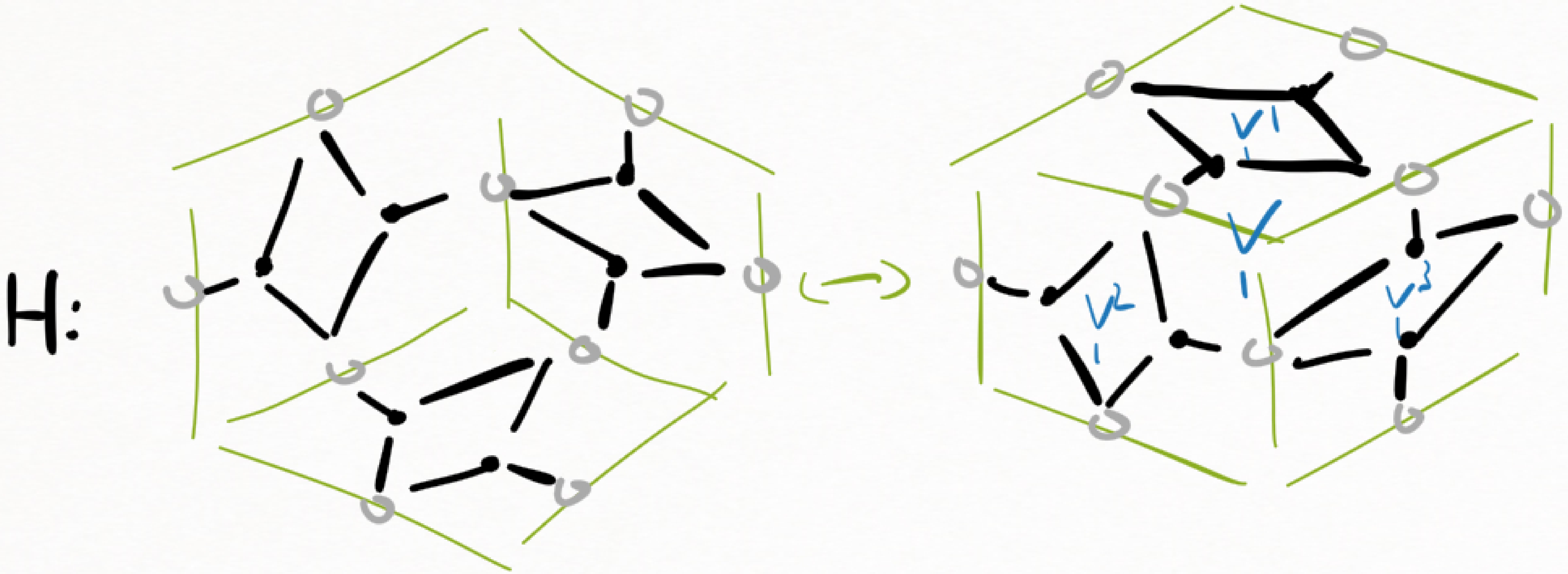
← Combinatorics that admit the Ising model



In Resistor subvariety iff

$X^1 \cdot X^2 \cdot X^3 = 1$

(In geometry: Koenigs nets)



In Ising subvariety iff

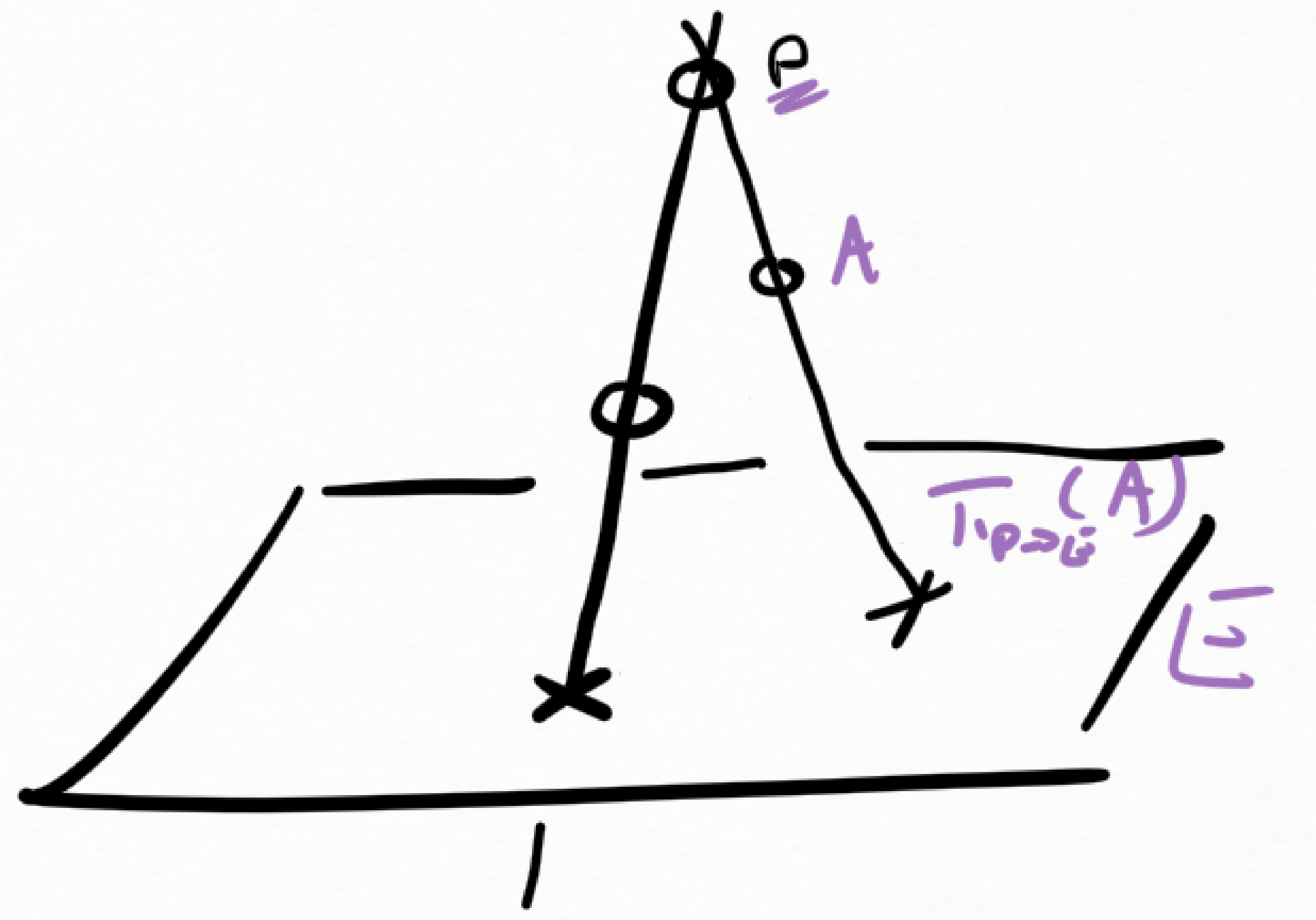
$$(Y)^2 (1+Y^1)(1+Y^2)(1+Y^3) = 1$$

((C-quadr lattices [D]))

Projections

- Projection:

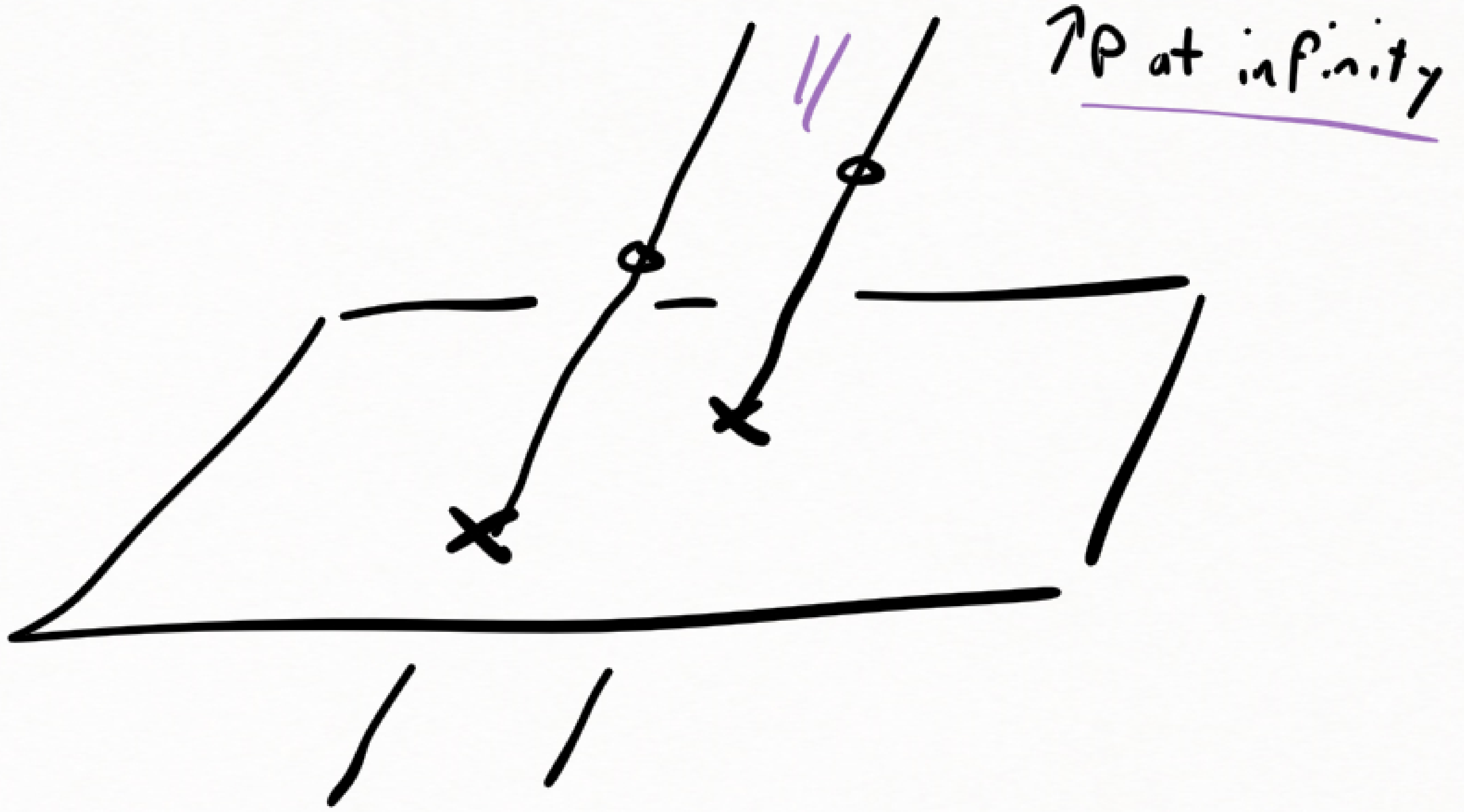
$$X(\pi_{p \to E}(T)) = X(T)$$



- Parallel projection:

$$Y(\pi_{p \to E}(T)) = Y(T)$$

if $p \in E^{\infty}$



s-embeddings

Th s-embeddings are projections of certain Q-nets to $\mathbb{CP}^1$

Th Y variables of s-embeddings are in the Ising subvariety

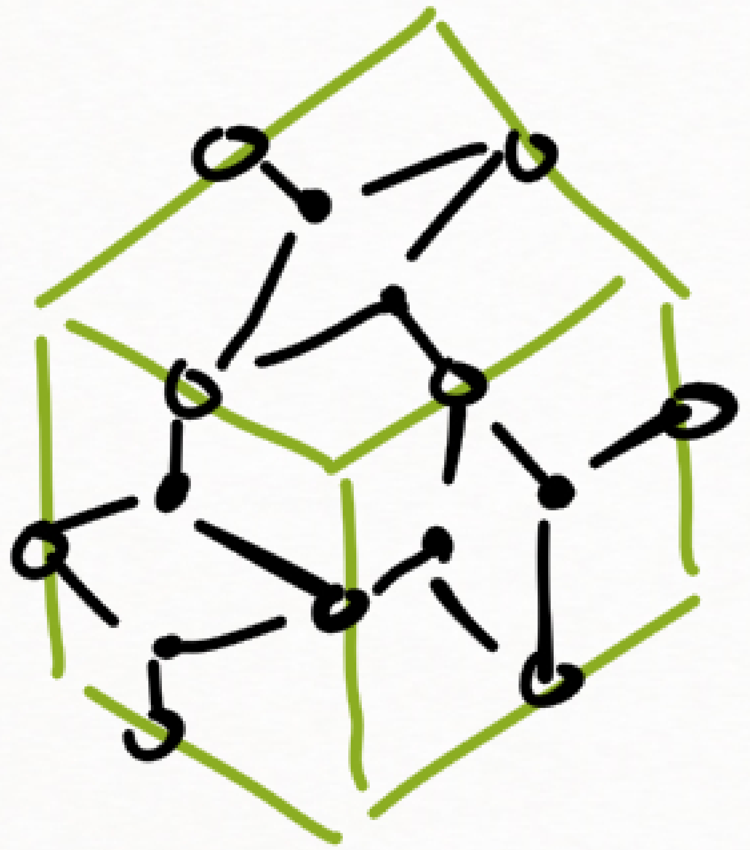
Not every projection of a Q-net $\subset$ I sub is an s-emb.

In $\mathbb{CP}^1$: $\supset \mathbb{C}$

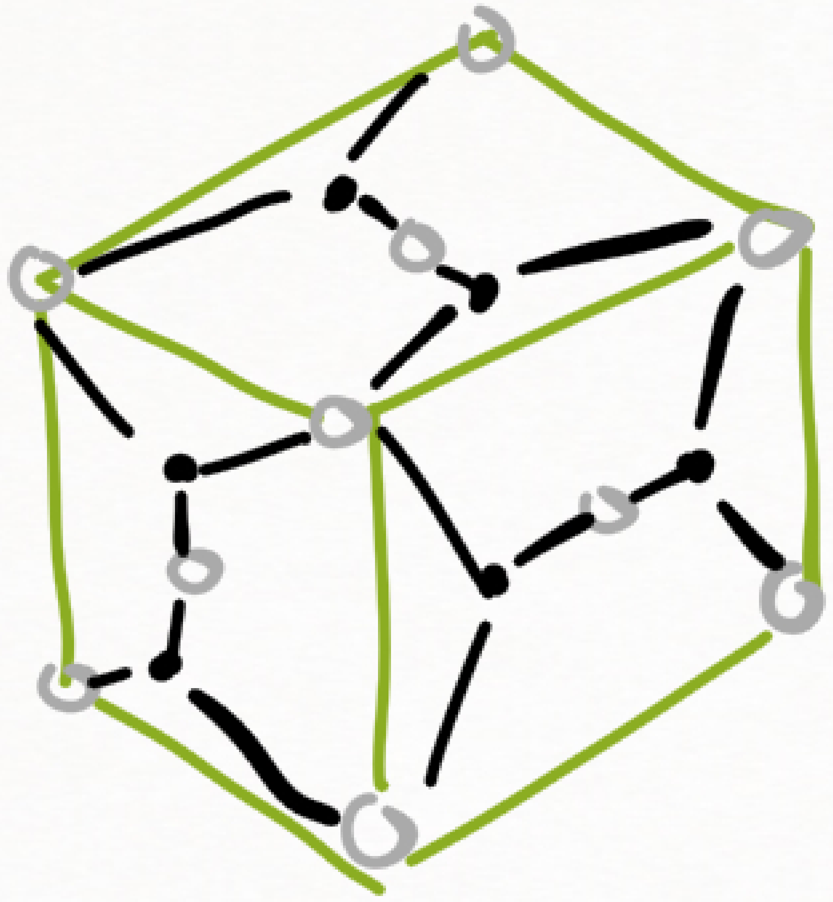
[Dima ↑]

Darboux maps

(optional)



← supports Ising model



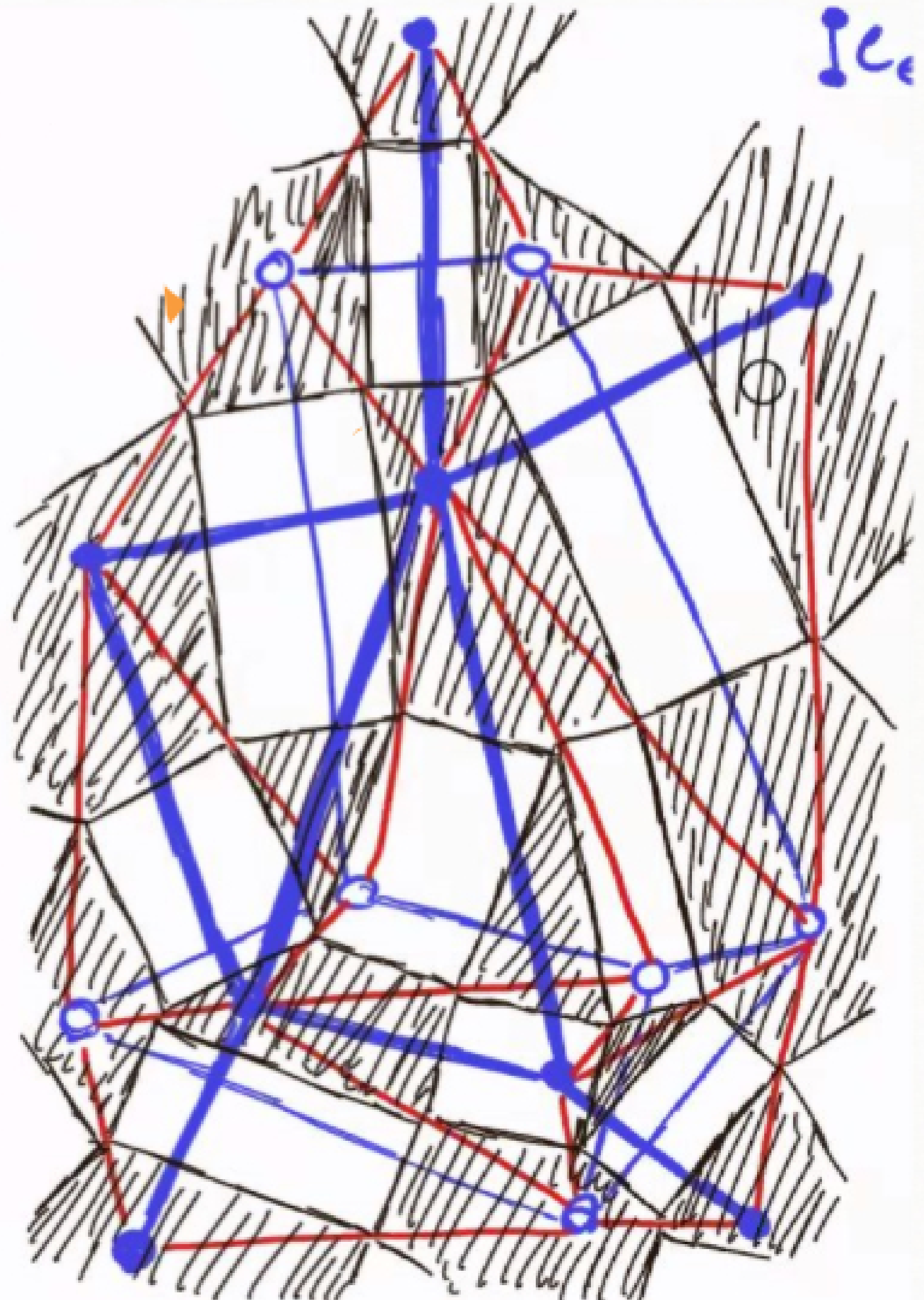
← supports Sp. tree model

[Schief]

Def: A map from the edges of a quadgraph such that every quad is colinear

Th: harmonic embeddings are projections of Darboux maps

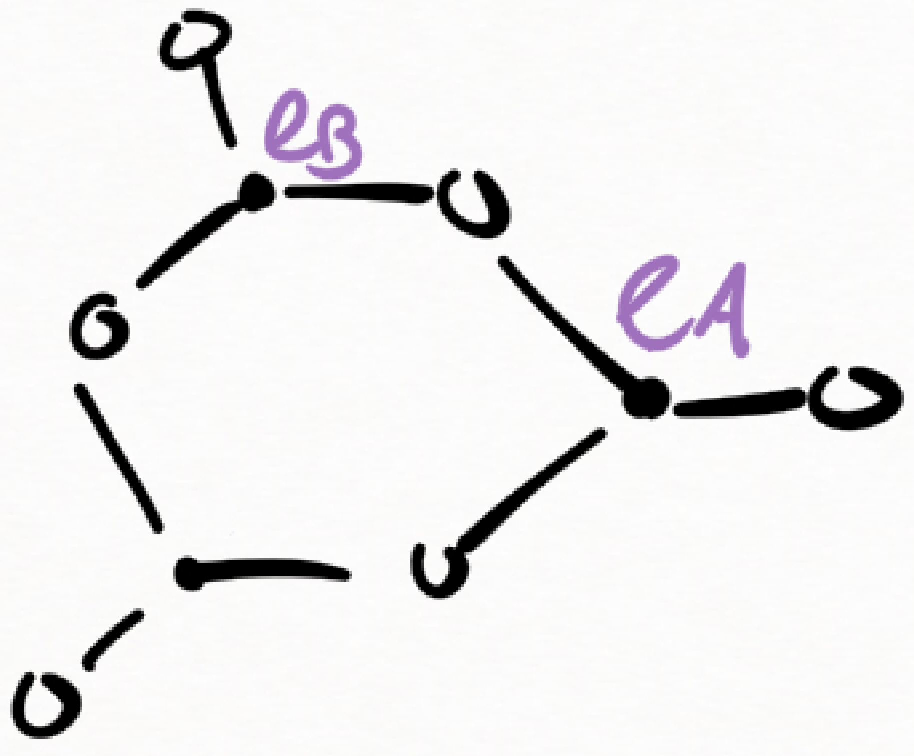
Th: The Y-variables of harm. emb. are in the Resistor subvariety



[Dima T]

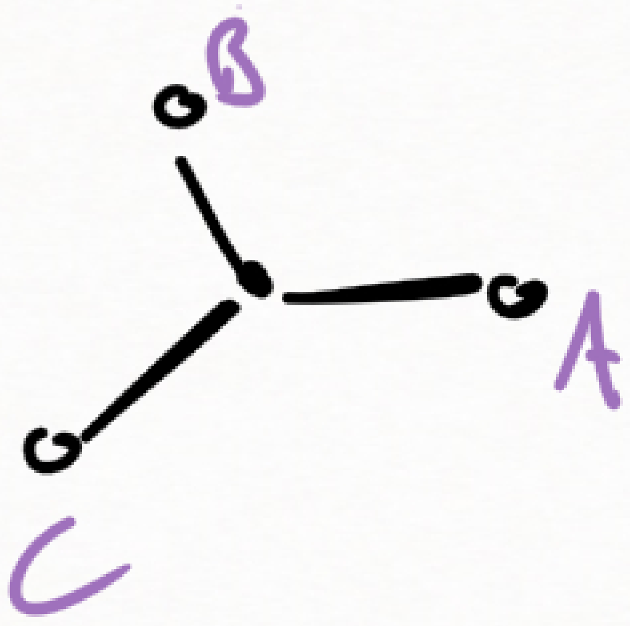
Sections

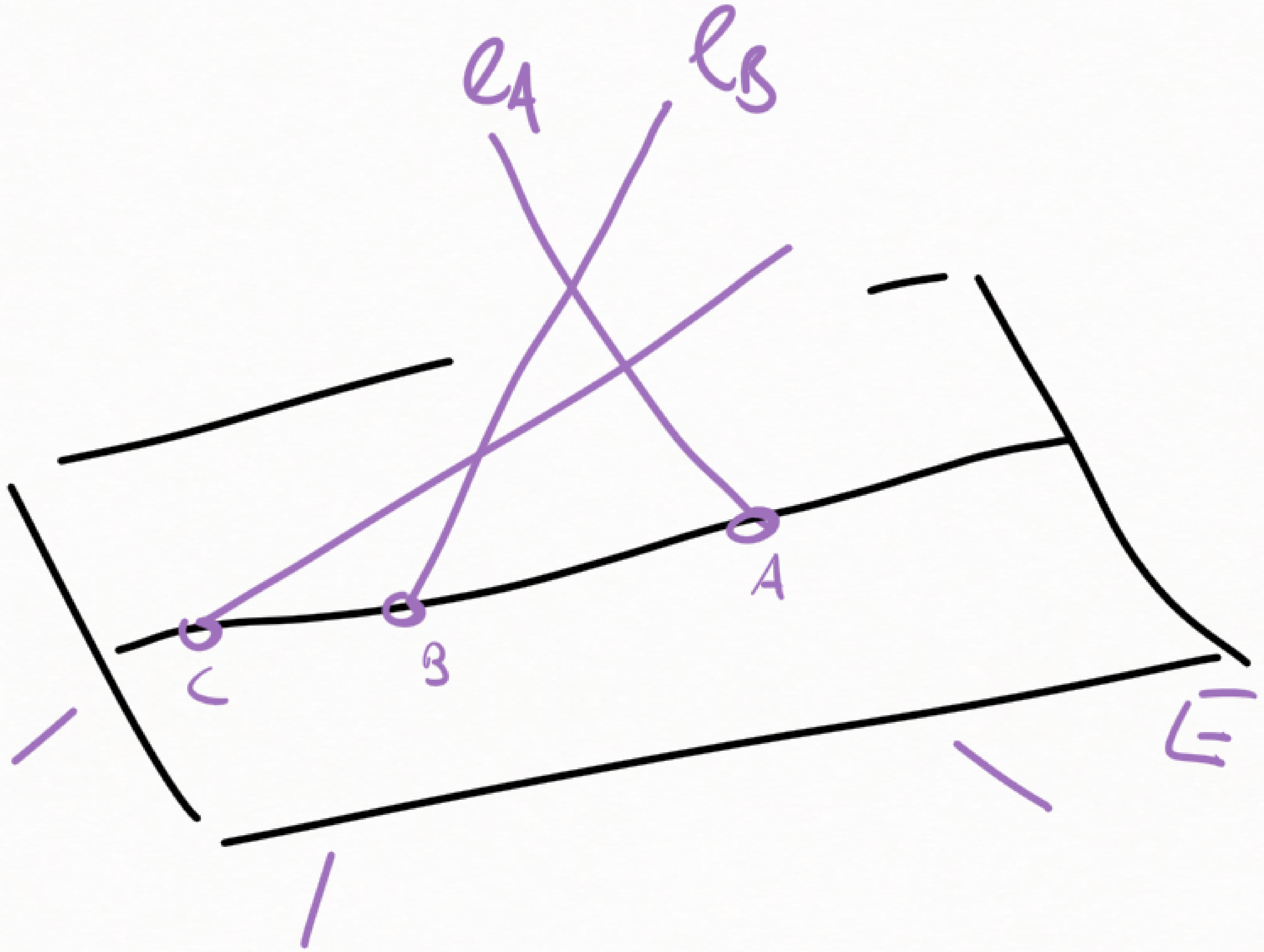
T



$$\downarrow \sigma_E$$

$\sigma_E(T)$





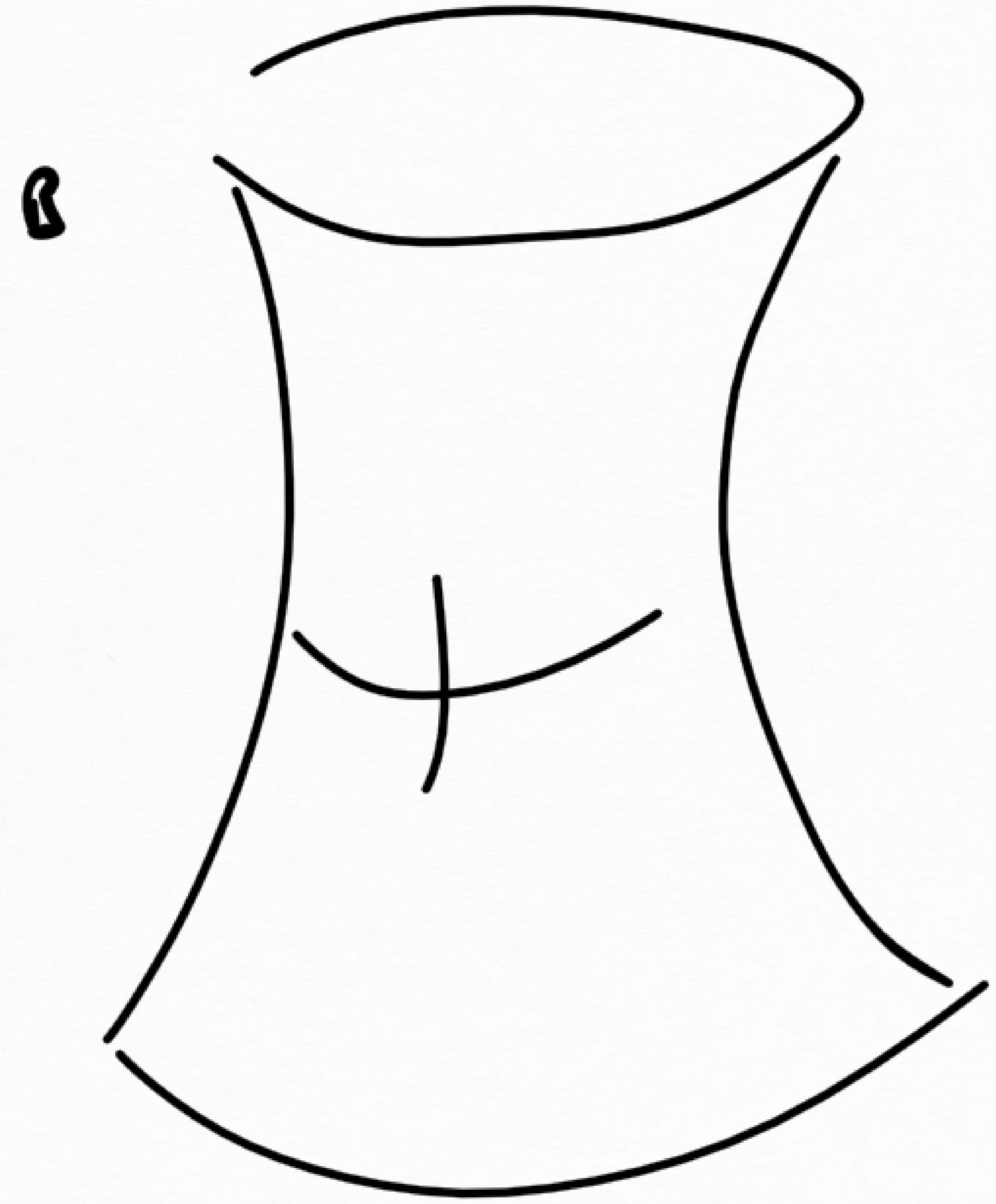
$$\boxed{\text{Th: } X(\sigma_{E_\infty}(T)) = Y(T)}$$

Problem: In $\mathbb{C}P^1$ doesnt work.

Remark:

Sections can be iterated.

Quadrics in $\mathbb{CP}^3$



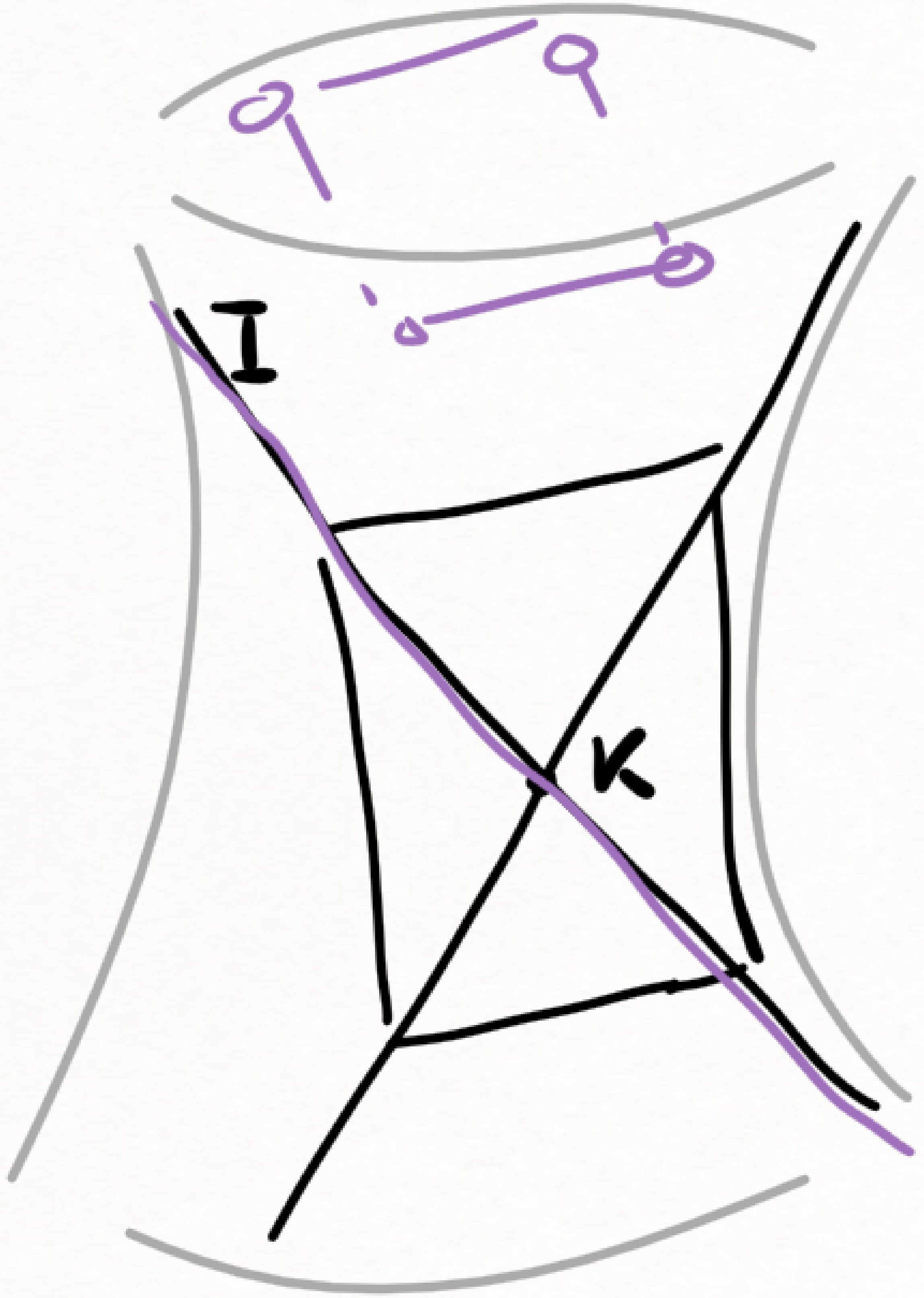
Def Let $b: \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$ be a symmetric bilinear form

$$B = \{[x] \in \mathbb{CP}^3 \mid b(x,x) = 0\}$$

is the quadric ass. to b

Def A line I is a generator if $I \subseteq B$

Q-nets in quadrics in $\mathbb{CP}^3$

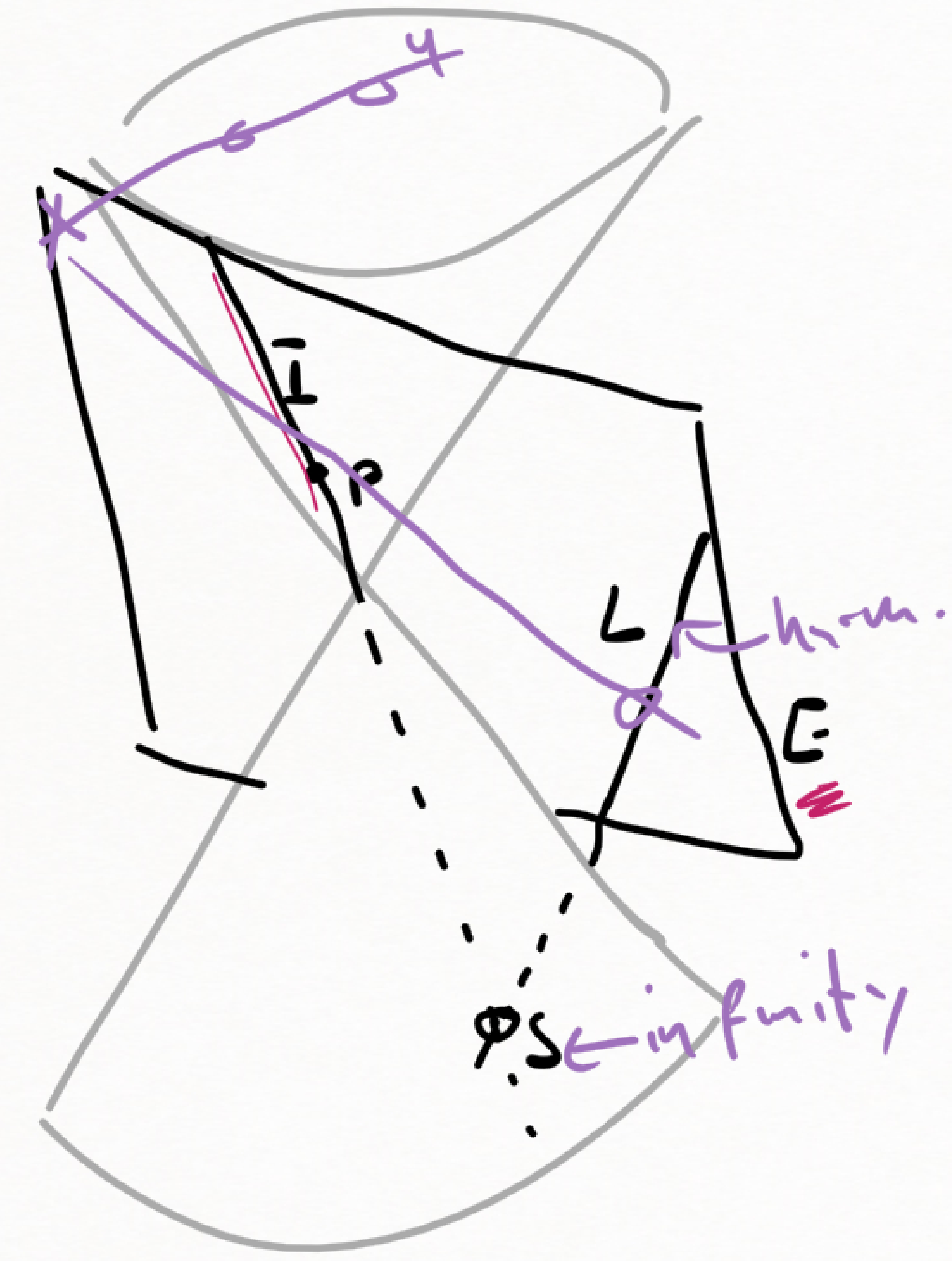


Th: Let $\mathcal{B}$ be a non-degenerate quadric and I a generator. Let q be a Q-net with points in $\mathcal{B}$.

Then $X(\sigma_I(q))$ is in the Resistor subvariety.

(There is a converse theorem)

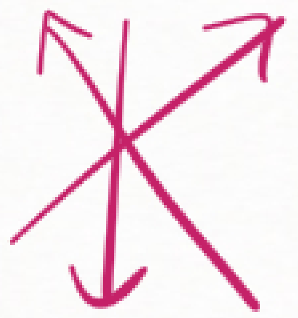
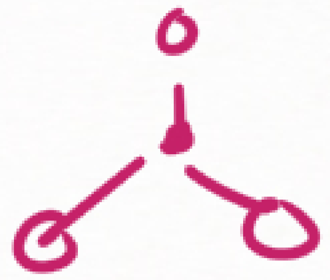
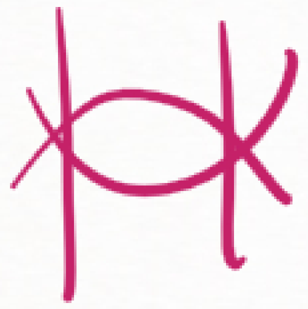
Harmonic embeddings are a special case

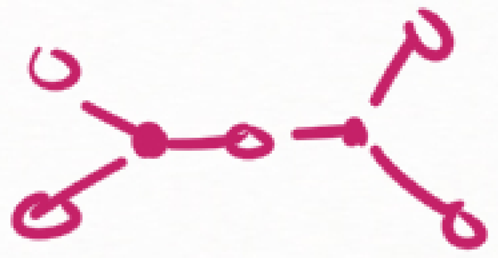
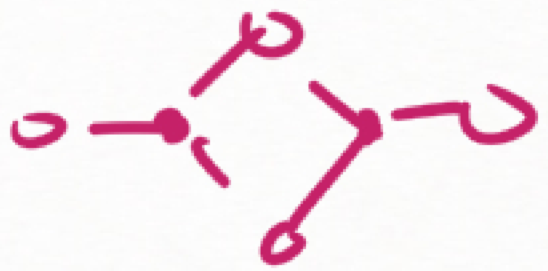


- B is a cone (degenerate quadric)
- E is a tangent plane
- Then there is a Q-net q in B, such that $\pi_{P \to L}(\sigma_E(q))$ is a given harm. emb. w.r.t. a chart of L with S at infinity.

(- The orthodiagonal quads are the result of projection $\pi_{Q \to E}(\pi_{P \to L}(q))$)

Conclusion

- TCD  → TCD map  → Proj. dimers → Aff. dimers
- 2-2 moves  →

Resplit 	invariant	change
spider move 	change	invariant

- Identify sp. tree / Ising subvarieties (connection discrete int. systems BKP/CKP)

  ↳ occur naturally in 1D sections of 3D objects ("linear") with quadratic constraints.

$\boxed{\text{The End}}$

Thanks to the organizers!