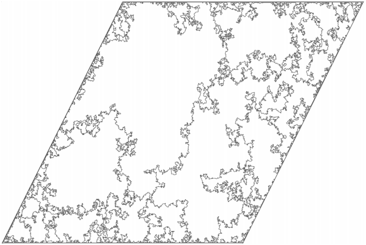
Dimers with free boundary conditions

Nathanaël Berestycki*
Universität Wien
with Marcin Lis (Vienna) and Wei Qian (Cambridge)



"Oberwolfach", November 2020

Free boundary model

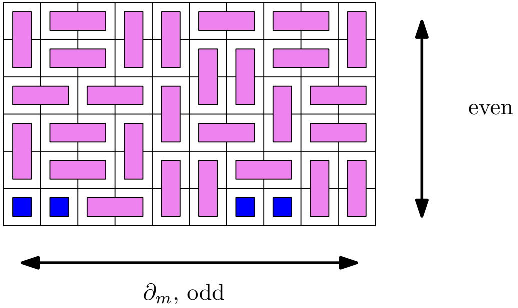
Allow monomers on part of the boundary, call it ∂_m.
Let $z > 0$ and define

$$\mathbb{P}(\mathbf{m}) \propto z^{\#\text{monomers}}.$$

Assumptions

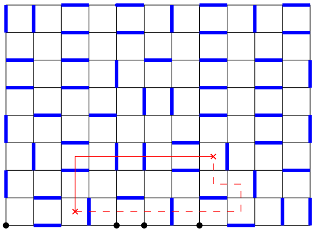
(1) G is dimerable (so partition function is > 0).
(2) $|\partial_m|$ is *odd* (a technical assumption).

Example:



Height function

Dual loops do not contain any monomers:



so height function still defined.

Main question

What is scaling limit of centered height function?
Conformal invariance?

Infinite volume limit

Theorem 1 (infinite volume limit)

Suppose monomer weights at the corners are given by

$$z' = \frac{z}{2} + \sqrt{1 + \frac{z^2}{4}}.$$

Then as $D_n \to \infty$, the free boundary dimer model has a limiting distribution on $\mathbb{Z}^2 \cap \mathbb{H} =$ upper half plane.

Scaling limit

Theorem 2 (scaling limit)

Let h^ε denote the height function on $\varepsilon\mathbb{Z}^2 \cap \mathbb{H}$ (viewed as distribution mod constant). Then

$$h^\varepsilon \to c\frac{1}{\sqrt{2\pi}}h_{\mathrm{GFF}}$$

in distribution, where h_{GFF} is a Gaussian free field with **Neumann boundary conditions** on $\partial\mathbb{H}$.

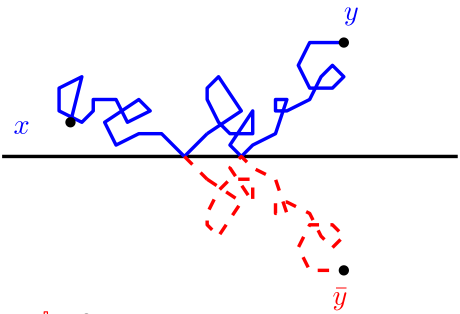
The factor c is in comparison with Temperleyan case...!

Note: first time the limit doesn't have Dirichlet conditions.

The Neumann GFF in $\mathbb{H}$

If $\int f(x)dx = 0$, then (h,f) has variance $\iint G(x,y)f(x)f(y)dxdy$ where

$$\mathbb{E}[h(x)h(y)] = G(x,y) = \int_0^\infty p_t(x,y)dt$$



$$p_t(x,y) = p_t^{\mathbb{C}}(x,y) + p_t^{\mathbb{C}}(x,\bar{y})$$

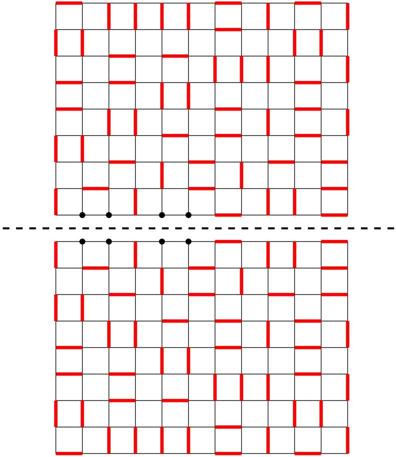
So

$$G(x,y) = -\log|x-y| - \log|x-\bar{y}|.$$

Only defined up to constant so G not unique.

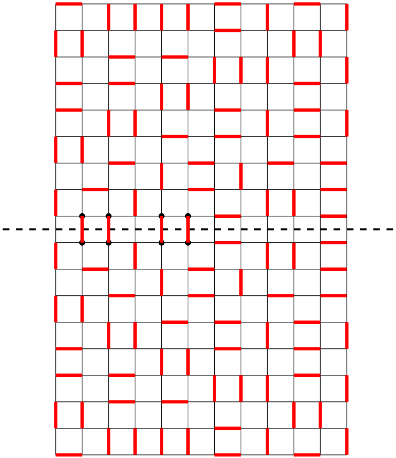
Reflection symmetry ($z = 1$)

Suppose $G \subset \mathbb{H}$ and $\partial_m \subset \mathbb{R}$. Then apply reflection:



Get a dimer configuration on G_{double}.

Height function



Let h_{double} = height function on G_{double}. Then *even*: $h_{\text{double}}(z) = h_{\text{double}}(\bar{z})$, and $h = h_{\text{double}}|_{\mathbb{H}}$.

Even/odd decomposition of full plane GFF

As $\varepsilon \to 0$, no conditioning, $h_{\text{double}} \to \frac{1}{\sqrt{2\pi}} h^{\mathbb{C}}_{\text{GFF}}$ (=full plane GFF)

Question

What is a (full plane) GFF conditioned to be even?

Fact:

$$h^{\mathbb{C}}_{\text{GFF}} = \frac{1}{\sqrt{2}}(h_{\text{odd}} + h_{\text{even}})$$

where $h_{\text{odd}}, h_{\text{even}}$ are independent Dirichlet/Neumann GFF.
So, conditioning to be even, $h^{\mathbb{C}}_{\text{GFF}} = \frac{1}{\sqrt{2}} h_{\text{even}}$.

This suggests $c = 1/\sqrt{2}$.

Microscopic effects

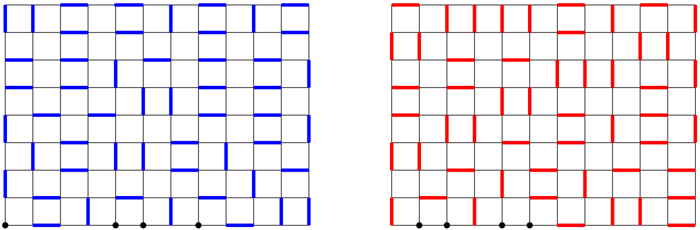
However the conditioning is very sensitive to microscopic details.

1d analogy: random walk bridge conditioned to be symmetric.

For SRW, scaling limit is again BM ($c = 1$).
But for Gaussian RW scaling limit is BM$/\sqrt{2}$ (so $c = 1/\sqrt{2}$.)

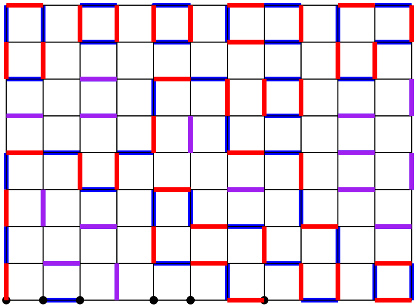
Double free boundary dimer

Superposition of two independent realisations of free boundary dimer model:



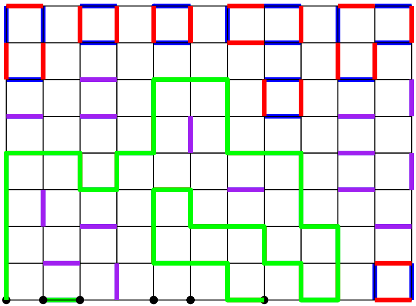
Double free boundary dimer

Superposition of two independent realisations of free boundary dimer model:



Double free boundary dimer

Superposition of two independent realisations of free boundary dimer model:



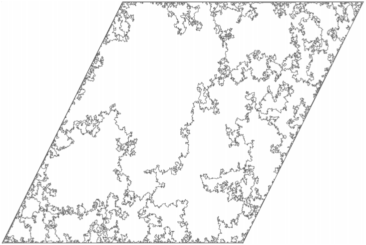
Get a collection of loops and boundary arcs.

Double free boundary dimer

Question:

In the scaling limit, what is the law of the loops and boundary arcs?

Natural object: **Arc Level Ensemble (ALE)** of Aru, Sepulveda and Werner $\approx$ 2015.



©B. Werness

Indeed, ALE = boundary touching level lines of Neumann GFF (Qian–Werner, CMP 2018).

Folding and superposition

Conjecture

Fold onto itself a full plane dimer model. Then boundary arcs converge to ALE. Loops converge to CLE_4 in remaining components.

At the discrete level, the height gap on either side of the curves corresponds to 2λ, same as ALE or CLE_4 in the continuum.

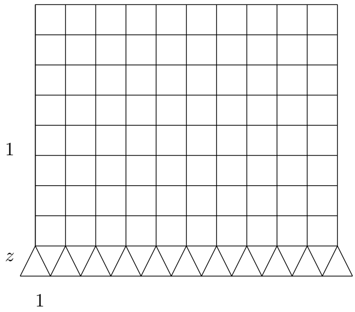
Free boundary case

Height gap corresponds to $2\lambda/c$. Yet there is no curve coupled with the GFF unless $c = 1$.

Sketch of proof of main result

Bijection to non-bipartite dimer

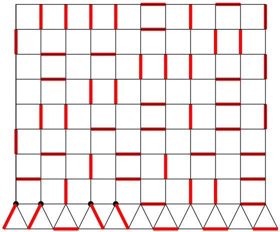
Giuliani, Jauslin, Lieb: Pfaffian formula for correlations.
In fact, bijection with dimer model:



Sketch of proof of main result

Bijection to non-bipartite dimer

Giuliani, Jauslin, Lieb: Pfaffian formula for correlations.
In fact, bijection with dimer model:



Lemma

If $|\partial_m|$ odd, and $\#$ monomers even, then unique way to associate dimer configuration on augmented graph.

Kasteleyn theory

Problem:

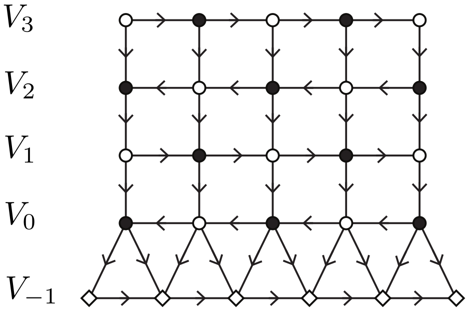
Graph becomes non-bipratite.

Kasteleyn theory

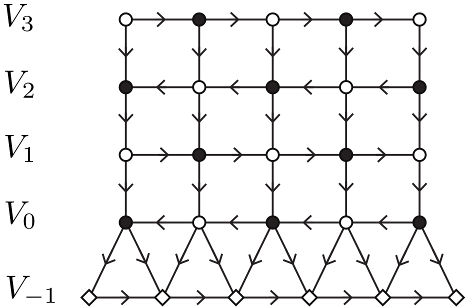
Kasteleyn orientation: going cclw on every face, odd number of clw arrows
Gauge transform: weight of every edge coming of a vertex v $\rightarrow \times\lambda_v$, with $\lambda_v \in \mathbb{C}, |\lambda_v| = 1$.
Kasteleyn matrix: $K(u, v) =$ signed weight of edge (u, v) (so K antisymmetric).

Then partition function $= |Pf(K)|$ and correlations are given by $Pf(K^{-1})$.

Kasteleyn orientation

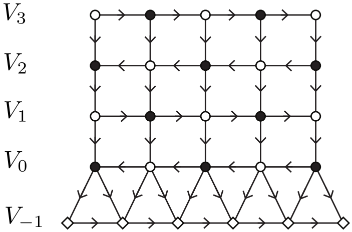


Gauge transform



Even rows multiplied by i:
$\implies$ each edge $e \in E_{\text{even}}$ multiplied by -1;
and each vertical edge has weight i.

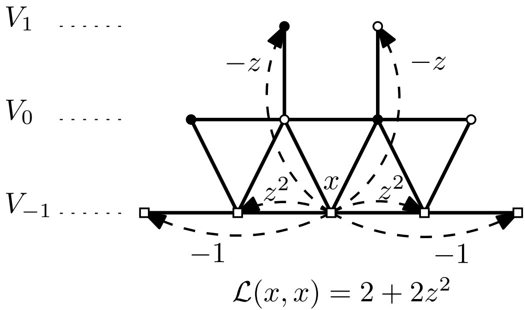
Kasteleyn matrix in bulk



Kenyon: consider $\mathcal{L} = K^* K$.
K = n.n. so $\mathcal{L}$ nonzero only from $W \to W$, $B \to B$.
Diagonal contributions vanish
So really $W_0 \to W_0, \ldots B_1 \to B_1$.
Then $\mathcal{L}$ = discrete Laplacian on each four sublattices.
Temperleyan b.c.: $\implies$ $\mathcal{L}$ has Dirichlet b.c. on B_0.

Kasteleyn matrix near monomers

At rows 1, 0, -1 the above analysis breaks down:



but $-\sum_{y \sim x} \mathcal{L}(x,y) = 2 - 2z^2 + 2z$.

Dealing with negative rates

Let $P(x,y) = -\mathcal{L}(x,y)/\mathcal{L}(x,x)$: can be signed, don't sum to 1...

Question

Can we still make sense of Green function?

If $\|P\| < 1$ then

$$\mathcal{L}^{-1}(x,y) = \frac{1}{\mathcal{L}(y,y)} \sum_{\text{path } \pi : x \to y} w(\pi)$$

where

$$w(\pi) = \prod_{(u,v) \in \pi} P(u,v).$$

Excursions

Paths still restricted even $\to$ even and odd $\to$ odd rows.
Decompose in excursions into V_{-1} or V_0.
Let $u, v \in V_{-1}$ and let $\pi : u \to v$. Parity fixed so

$$w(\pi) = (-1)^{v-u} \prod_{(x,y)\in\pi} p_{x,y}$$

where

$$p_{i,i\pm1} = \frac{z^2}{2+2z^2} =: 1/2 - p, \qquad p_{i,i\pm2} = \frac{1}{2+2z^2} =: p.$$

$\implies$ an honest RW if $V_{-1} \simeq \mathbb{Z}$. This corresponds to

$$z' = (z/2) + \sqrt{1 + z^2/4}.$$

Excursions and potential kernel

So

$$\sum_{\pi:u\to v;\pi\subset V_{-1}} w(\pi) = (-1)^{v-u} g_{u,v} \qquad (\star)$$

where $g_{x,y} = 1d$ Green function.

Question

What does this correspond to when we consider paths $V_1 \to V_1$?

Let $u, v \in V_1$. We associate vertices $u_\pm$ and $v_\pm$ in V_{-1}.

Sum $(\star)$ over $u_\bullet \in \{u_-, u_+\}, v_\bullet \in \{v_-, v_+\}$:

$$\sum_{\pi:u\to v;\pi\subset V_{-1}} w(\pi) = C_z(-1)^k(2a_k - a_{k+1} - a_{k-1})$$

where a_k = Potential kernel of 1d walk; $k = \mathrm{Re}(v-u)$.

The miracle

Lemma

$(-1)^k \Delta a_k \geq 0$ for all $k \in \mathbb{Z}$
Moreover $\sum_{k\in\mathbb{Z}} C_z(-1)^k(2a_k - a_{k+1} - a_{k-1}) = 1$.

Gives an effective random walk on $V_1 \cup V_3 \cup \ldots \simeq \mathbb{H}$!
Schur complement formula: random walk representation for K^{-1}

Theorem

Choosing z' as above, for fixed finite domain,

$$K^{-1} = \mathcal{L}^{-1} K^*,$$

where $\mathcal{L}^{-1}$ is the Green function of the effective walk.

Towards scaling limit

Lemma

$(-1)^k \Delta a_k$ decays exp. fast as $k \to \infty$.

So: effective walk jumps on the boundary, but exponential tails.
Scaling limit: BM with reflection!.

Notice that $\mathcal{L}^{-1}(u, v)$ not restricted to $B \to B, W \to W$:
However paths must go through boundary !

Thanks for your attention!

Pfaffian calculation

Eg: $e = (w, b); e' = (w', b')$

$$\mathbb{P}(e, e' \in \mathbf{m}) = \mathrm{Pf}\begin{pmatrix} 0 & K^{-1}(w,b) & K^{-1}(w,w') & K^{-1}(w,b') \\ & 0 & K^{-1}(b,w') & K^{-1}(b,b') \\ & & 0 & K^{-1}(w',b') \\ & & & 0 \end{pmatrix}$$

$$= \overbrace{K^{-1}(w,b)}^{\mathbb{P}(e\in\mathbf{m})}\overbrace{K^{-1}(w',b')}^{\mathbb{P}(e'\in\mathbf{m})} + K^{-1}(b,w')K^{-1}(w,b') - K^{-1}(w,w')K^{-1}(b,b')$$

so

$$\mathrm{Cov}(1_{e\in\mathbf{m}}; 1_{e'\in\mathbf{m}}) = K^{-1}(b,w')K^{-1}(w,b') - K^{-1}(w,w')K^{-1}(b,b')$$

Leads to scaling limit eventually...! □

Proof of oscillations

a_k solves a recurrence relation of order four.
Also by general theory [e.g. Lawler–Limic]:

$$a_x \sim \frac{|x|}{\sigma^2} \text{ as } |x| \to \infty$$

Hence

$$a_x = \frac{|x|}{\sigma^2} + A + B\gamma^{|x|}$$

where

$$1 = (1/2 - p)(\gamma + \gamma^{-1}) + p(\gamma^2 + \gamma^{-2}).$$

Hence let $s = \gamma + \gamma^{-1}$

$$1 = (1/2 - p)s + p(s^2 - 2).$$

Can solve s so $s = 2$ or $s = -1 - 1/(2p)$.
This implies $\gamma \in (-1, 0)$ so oscillations.