s-/t-/p-embeddings (1/3)

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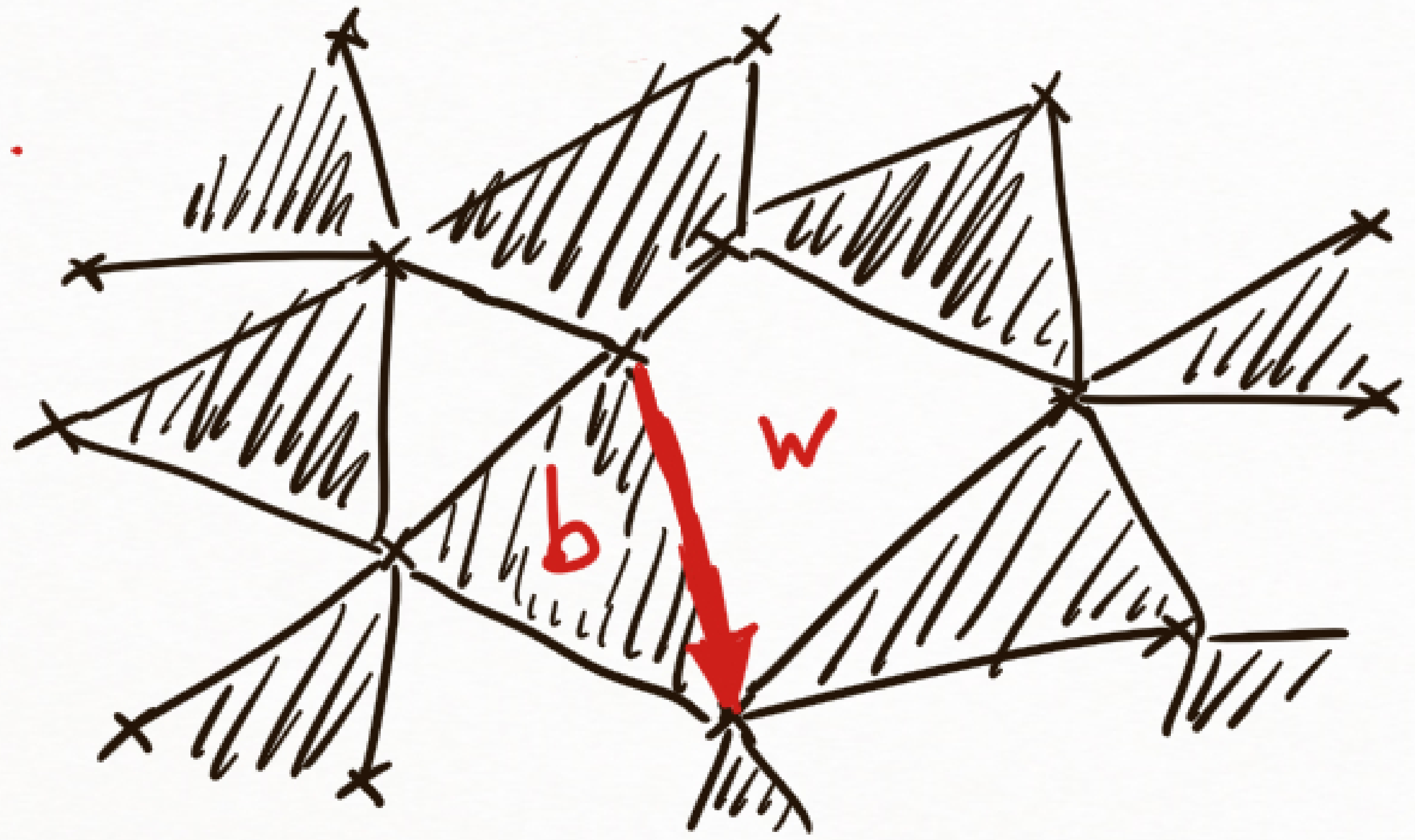
mini-workshop @ Zoom

November 17, 2020

DIMERS: t-embeddings [CLR] <=> Coulomb gauges [KLRR]

$\mathcal{T}$ = embedding of G^* into $\mathbb{C}$ s.t.



① lengths are gauge equiv. to (given) dimer weights

② angles at (inner) vertices are balanced : $\sum \text{white} = \pi = \sum \text{black}$

G - weighted bipartite graph

$K(b,w)$ - real-signed Kasteleyn

Let $g^\circ \in \mathbb{C}^W : Kg^\circ = 0$ (locally)

$g^\bullet \in \mathbb{C}^B : g^\bullet K = 0$

Define $\mathcal{K}(b,w) := g^\bullet(b) K(b,w) g^\circ(w)$

(gauge equivalent weights)

and $d\mathcal{T}((bw)^*) := \mathcal{K}(b,w)$

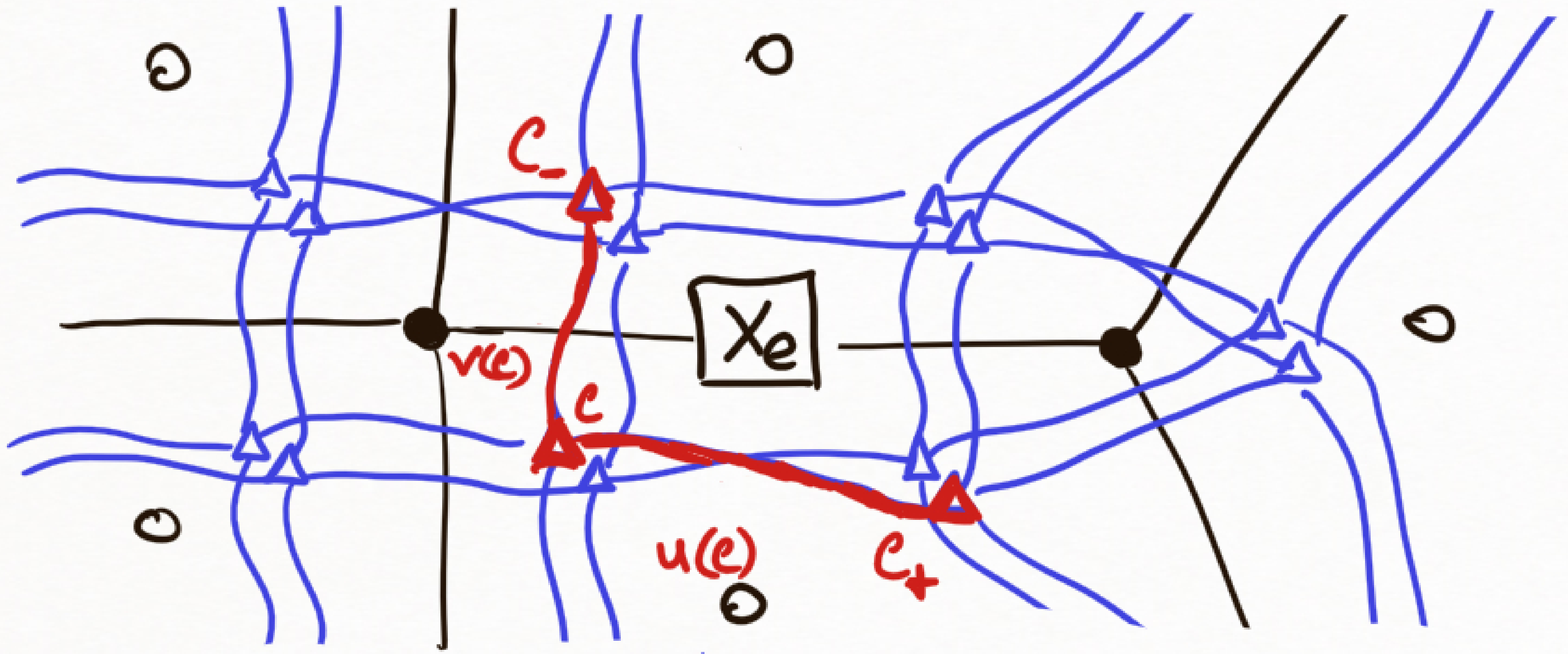
(well-def. on G^* if $Kg^\circ = 0 = g^\bullet K$)

Rem: ② <=> Kasteleyn sign condition

Ising vs Bipartite Dimers : s-embeddings $\longleftrightarrow$ t-embeddings

Fermionic observables (Ising)

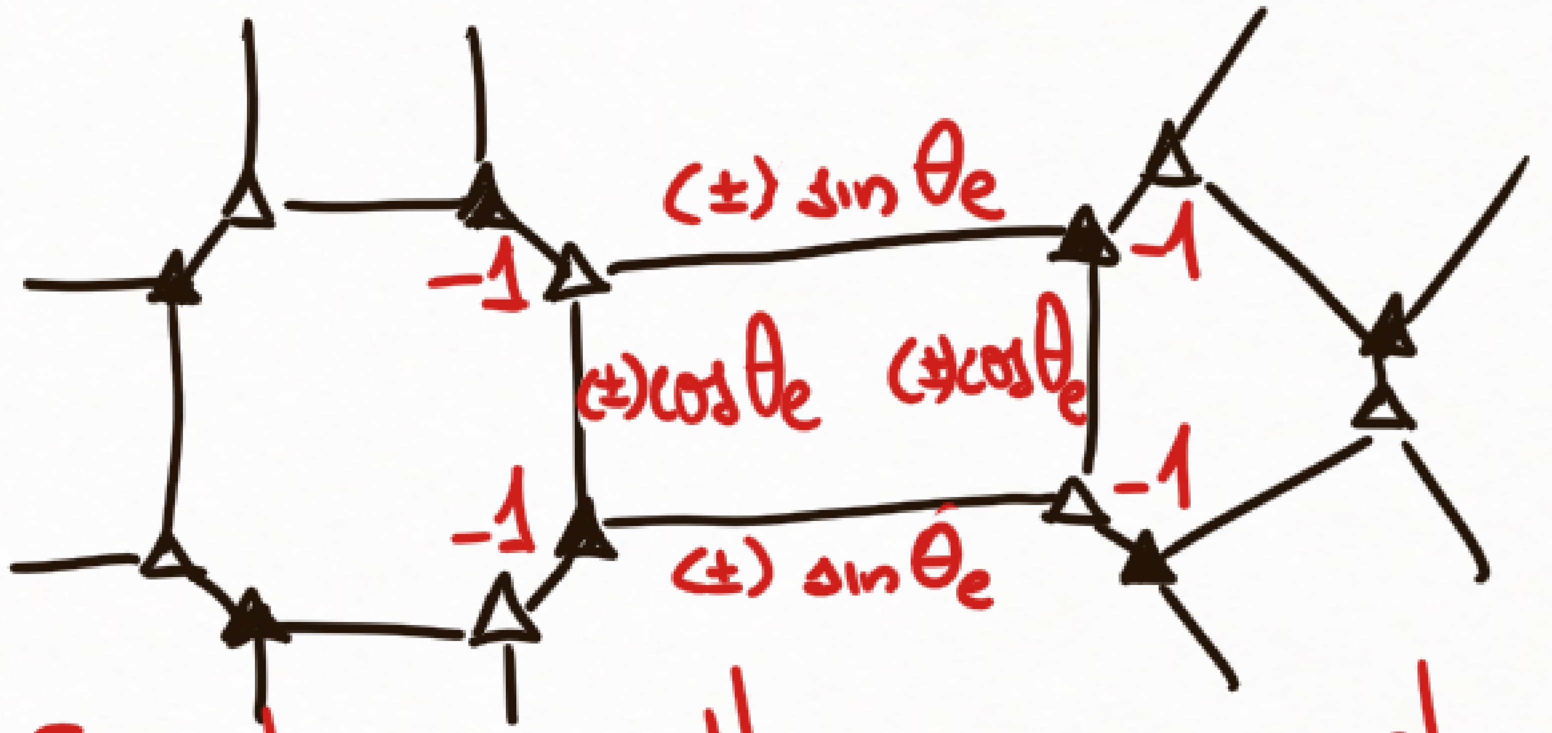
(parametrization of interactions $x_e = \tan \frac{1}{2}\theta_e$)



$$X(c) := \langle \sigma_{u(e)} \mu_{v(e)} \ldots \rangle$$

[Kadanoff - Ceva spin-disorder does not require an embedding]

Inverse Kasteleyn matrix (dimers)



[sections of the double cover $\Longleftrightarrow$ choices of $(\pm)$ Kasteleyn signs]

"Propagation equation"

$$X(c) = X(c_-)\cos\theta_e + X(c_+)\sin\theta_e$$

$$\Longleftrightarrow \; KX = 0 \; \Longleftrightarrow \; XK = 0$$

(locally)

ISING : s-embeddings

$\mathbb{Z}^2 \subset$ isoradial (rhombi)

$\cap$

Lis' circle patterns (kites)

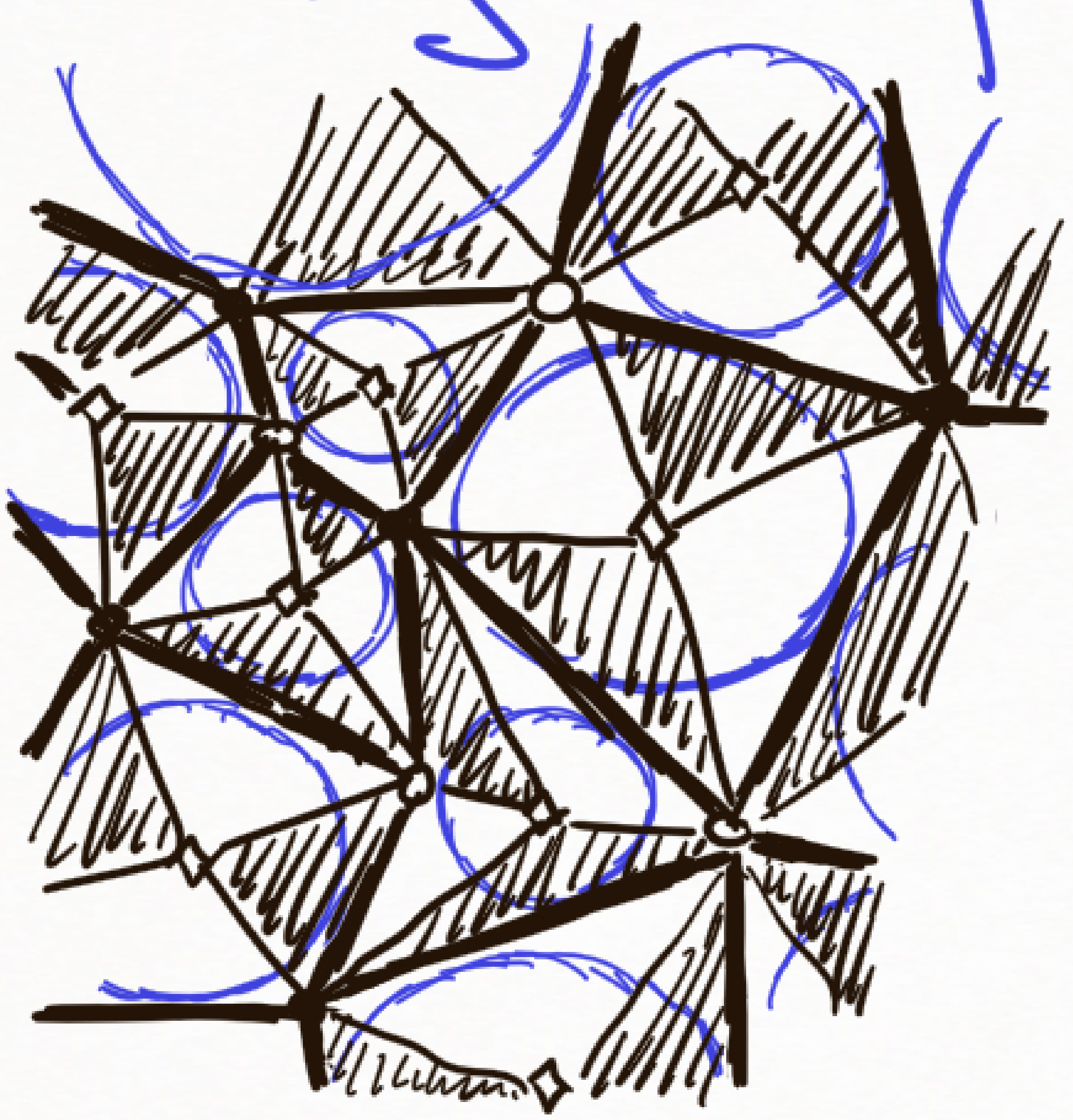
$\cap$

tangential quads [non-convex OK]

$\cap$

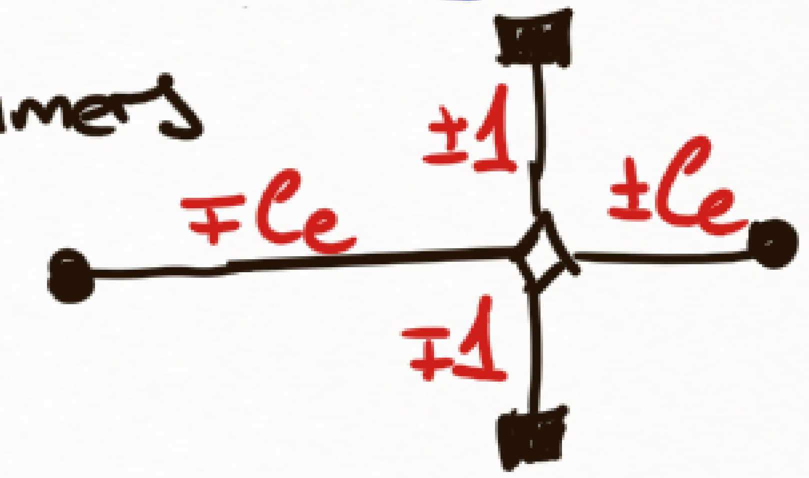
t-embeddings

[coherent w/ Ising ↔ dimers correspondence]

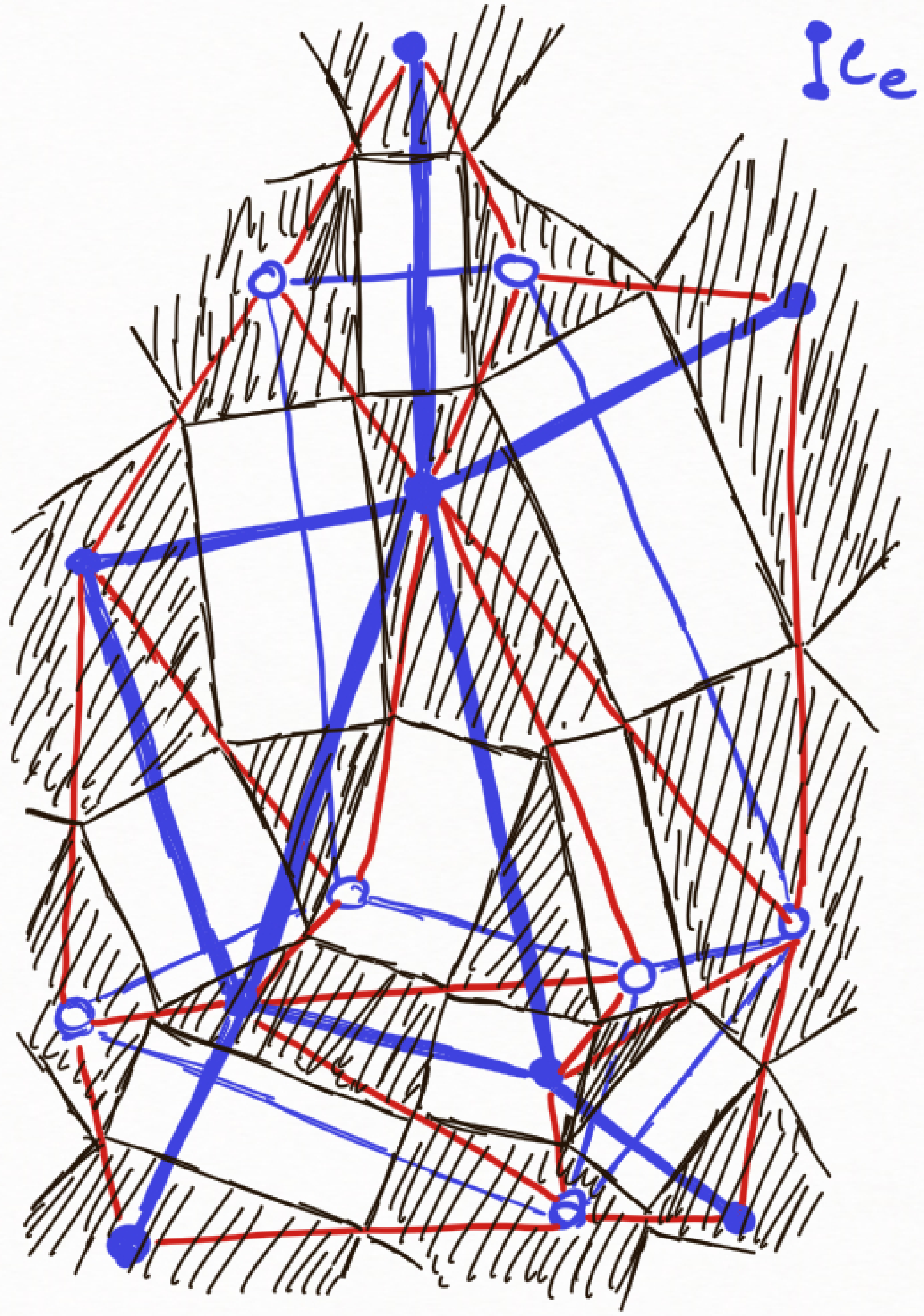


HARM : bi-orthogonal embeddings

conductances c_e ⇝ dimers



(white faces = rectangles)



Rem: origami map (see below)

Ising : 1D

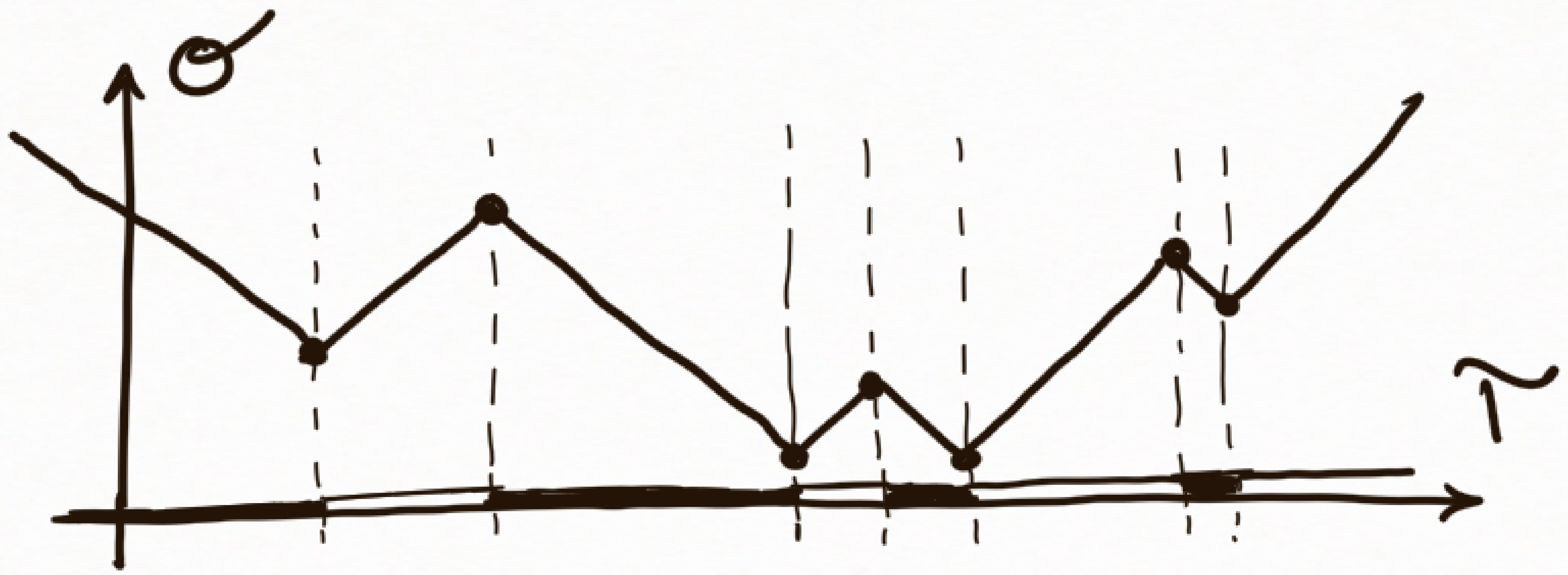
Harm : 0D

ORIGAMI MAP : "fold the plane along each edge"

[angle condition $\sum$ black $= \pi = \sum$ white $\Rightarrow$ local consistency]

[!] One could / should view t-embeddings as $(T, \mathcal{O}) \in \mathbb{R}^{2+2}$ (and s-emb. as $(\mathcal{S}, Q) \in \mathbb{R}^{2+1}$)

toy example in (1+1)D



Formally : a piece-wise linear map :

$d\mathcal{O} = \eta_w^2 \, dz$ on white faces

$d\mathcal{O} = \bar{\eta}_b^2 \, d\bar{z}$ on black faces

where $|\eta_w| = |\eta_b| = 1$

and $dT/|dT| = \bar{\eta}_b \bar{\eta}_w$

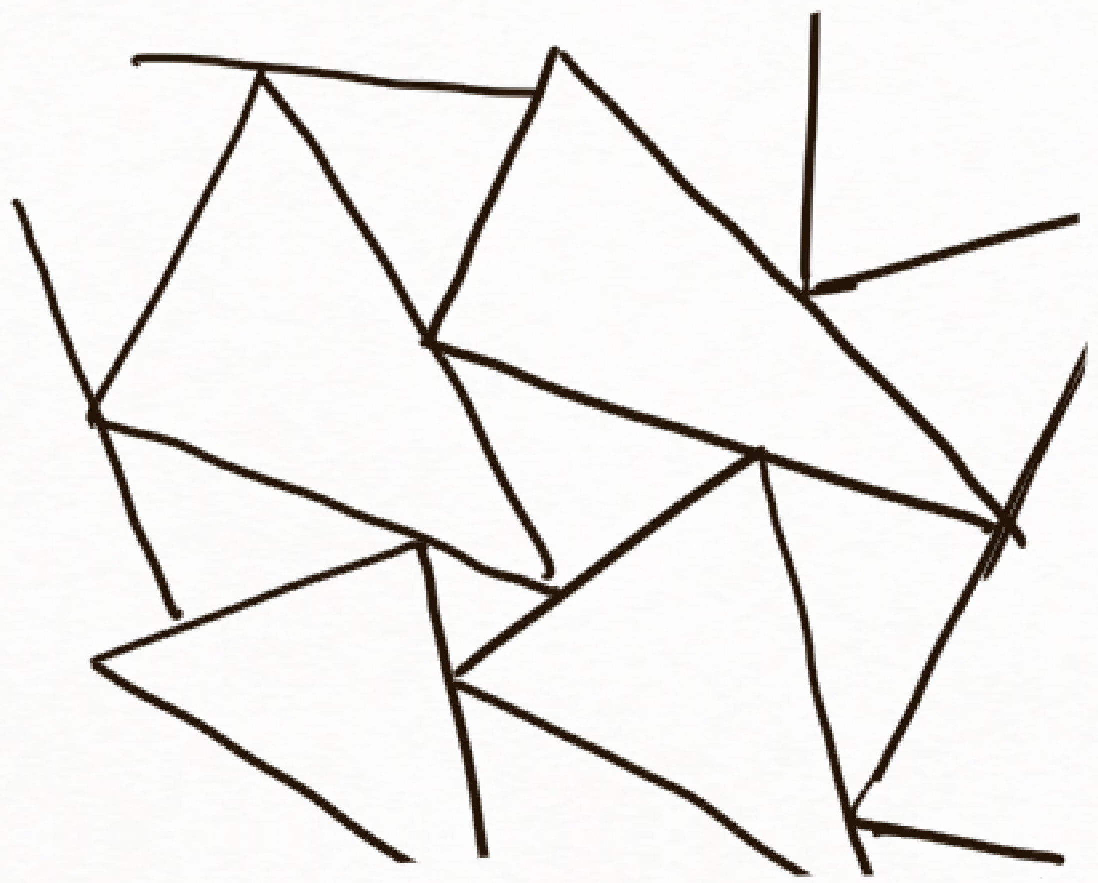
[$\Leftrightarrow$ $g^{\bullet}(b) \in \bar{\eta}_b \mathbb{R}$
gauge $g^{\circ}(w) \in \bar{\eta}_w \mathbb{R}$]

Note that $|\mathcal{O}(x) - \mathcal{O}(y)| \le |T(x) - T(y)|$ $\rightsquigarrow$ space-like surfaces

t-/s-embeddings + origami = T-/S-graphs

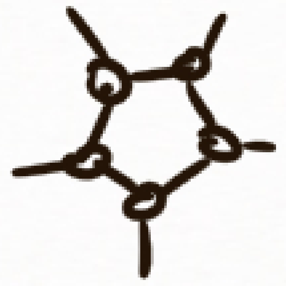
[see an example in GeoGebra]

T-graphs carry natural directed random walks



HARM:

$T+\mathcal{O}$: 

$T-\mathcal{O}$: 

usual RW

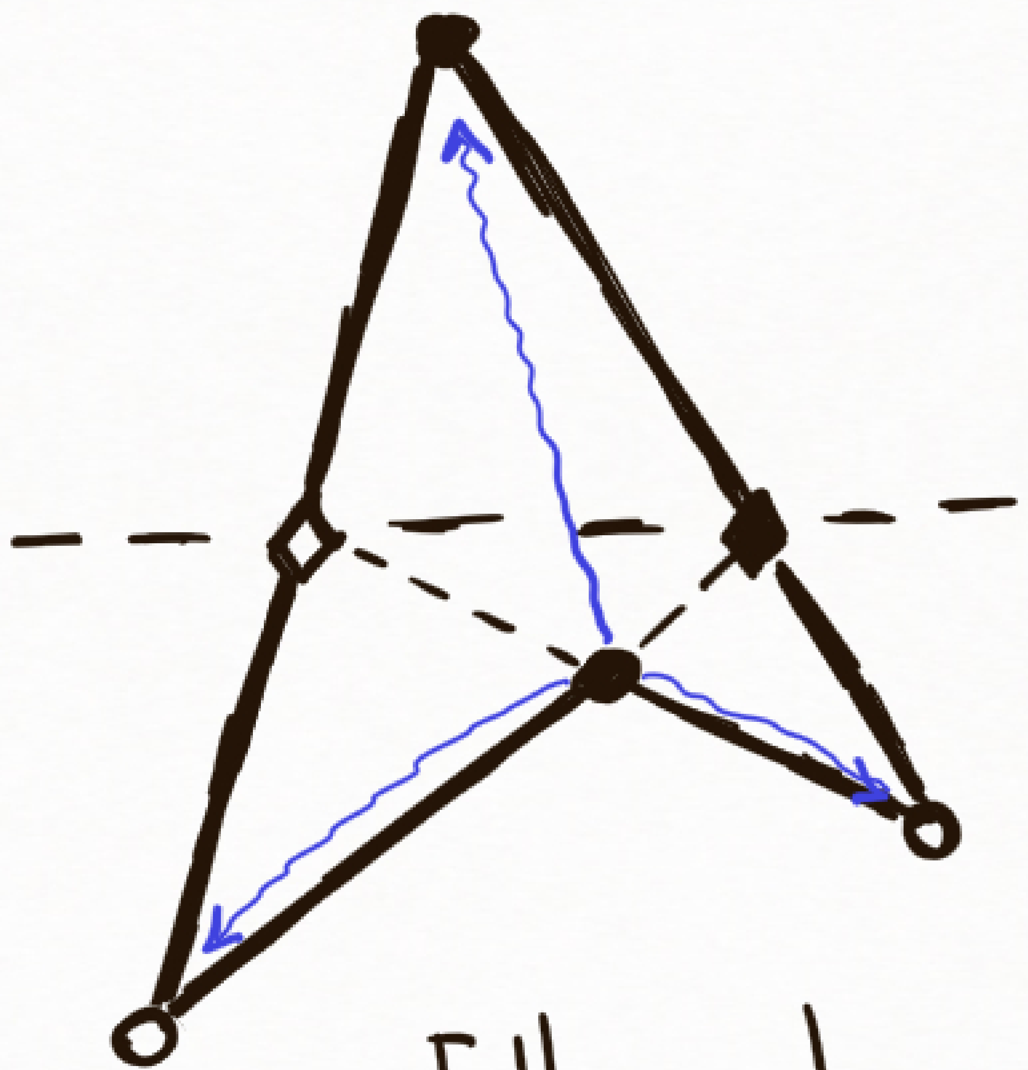
Fact: t-holomorphic functions: $\bar{\alpha} \times$ (gradients of $\mathbb{R}$-valued harm. functions on $T+\alpha^2\mathcal{O}$)

this notion does not depend on α

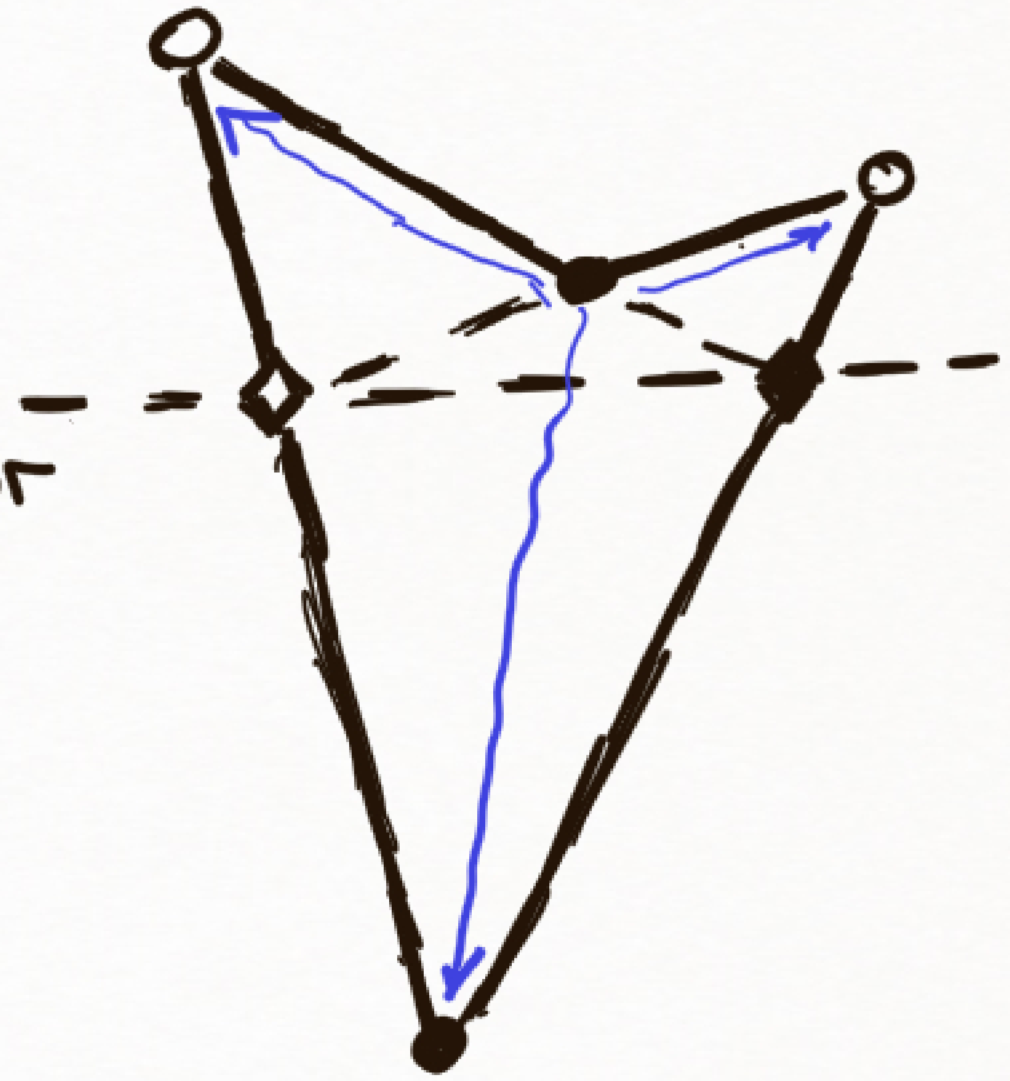
T-graphs: Kenyon(-Sheffield-Wilson)

S-graphs: just a name

(e.g., quads in $S-Q$ are



or



[the direction of -◇--◆- ↳ α in $S+\alpha^2 Q$

Periodic case ([KLRR], see also [Aff])

periodic t-embeddings $\xleftrightarrow[\text{rot., scaling and conj.}]{\text{up to}}$ inner pts of the amoeba $\begin{bmatrix} g^\circ - (\lambda_1,\lambda_2)\text{-periodic} \\ g^\bullet - (\frac{1}{\lambda_1},\frac{1}{\lambda_2})\text{-periodic} \end{bmatrix}$

Thm ([KLRR]): such gauges indeed define an embedding (i.e., no self-intersections)

Remark: this is slightly more involved at real nodes of the spectral curve:

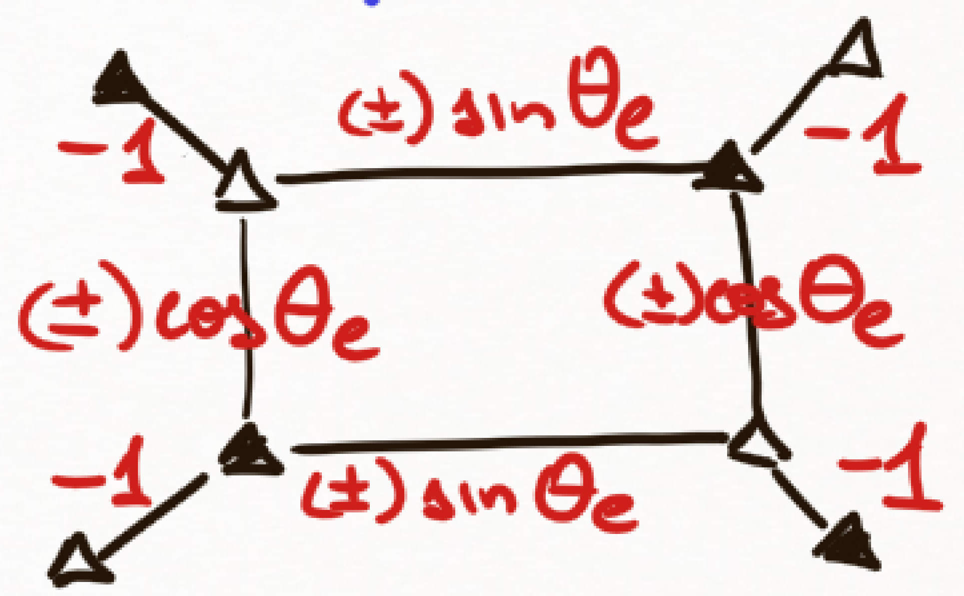
① Unique choice of $g^\bullet, g^\circ$ ⇝ periodic "origami map" $\mathcal{O}$

② $g^\bullet \mapsto \dfrac{g^\bullet + k^\bullet \overline{g}^\circ}{\sqrt{1-|k^\bullet|^2}}$

$g^\circ \mapsto \dfrac{g^\circ + k^\circ \overline{g}^\circ}{\sqrt{1-|k^\circ|^2}}$ ⇝ Lorentz-isometry of $(\mathcal{T},\mathcal{O}) \in \mathbb{R}^{2+2}$

ISING: $\lambda_1=\lambda_2=1$

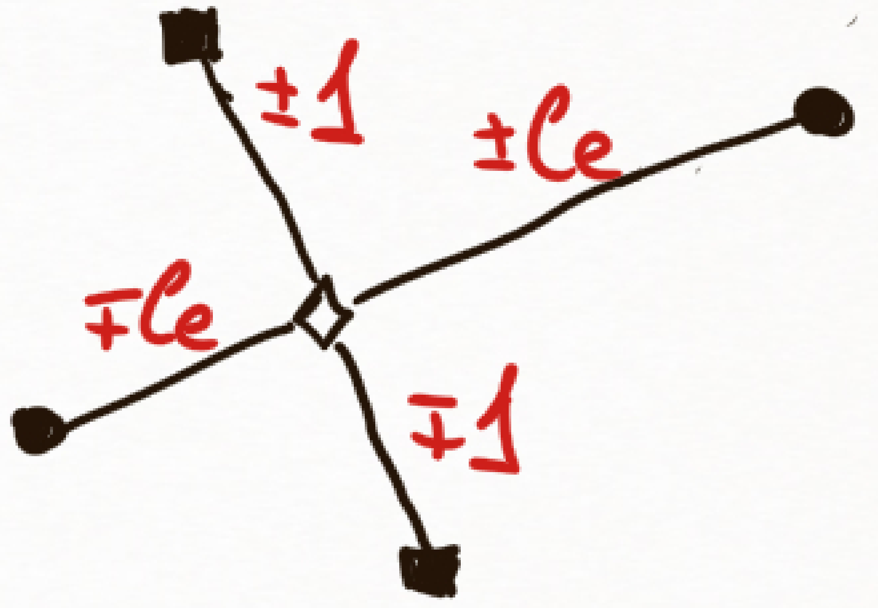
interested in $g^\blacktriangle = g^\triangle$



HARM: $\lambda_1=\lambda_2=1$

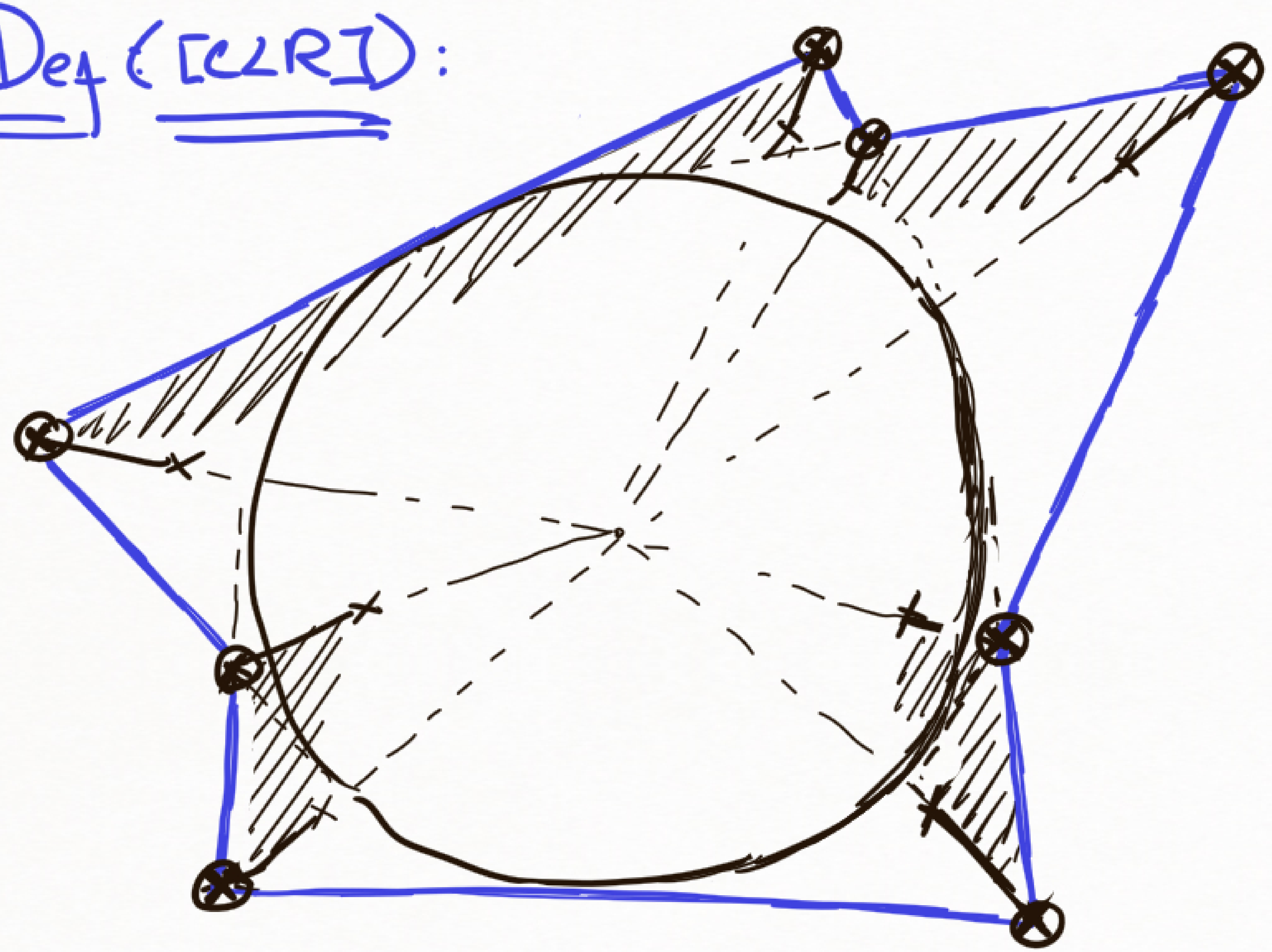
interested in

$g^\bullet = 1$, $g^\blacksquare = i$



FINITE BIPARTITE GRAPHS : P-EMBEDDINGS (= perfect t-emb.)

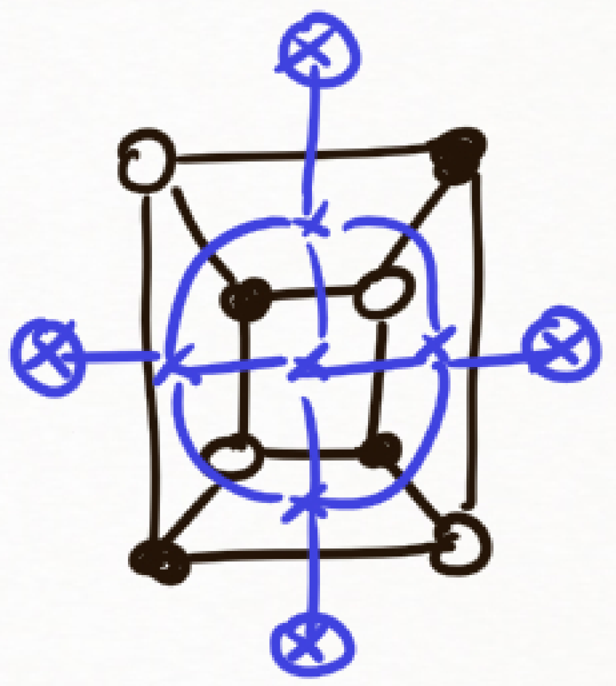
Def ([KLRR]):



Quartic equations on $\operatorname{Re}g^\bullet, \operatorname{Im}g^\bullet$
$\operatorname{Re}g^\circ, \operatorname{Im}g^\circ$

Lemma: no overlaps

① Outer face of the "augmented dual" is a tangential polygon

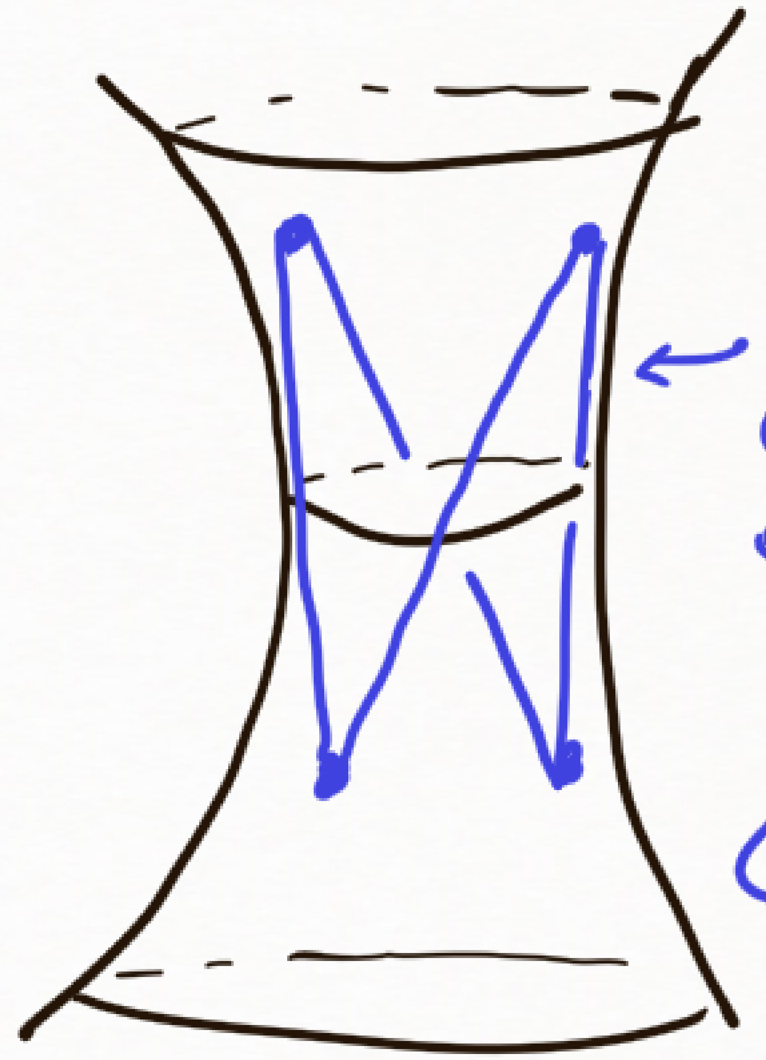


② Outgoing edges = bisectors

? EXISTENCE ?

? UNIQUENESS ?

(up to Lorentz isometries $g \mapsto \frac{g + k\bar{g}}{\sqrt{1-|k|^2}}$)



ROADMAP: "abstract" graphs ⤳ s/t-embedding ⤳ massive holomorphicity

- Weighted planar graphs (maps) carrying [Ising / bipartite dimers]
  → fermionic observables satisfy a simple local ($\mathbb{R}$-linear) relation (*)

- Choice of a "distinguished" ($\mathbb{C}$-valued) solution(s) of (*) → S-/T-embedding of G (or G^*) into $\mathbb{C}$ or, better, into Minkowski $\mathbb{R}^{2+1}$ or $\mathbb{R}^{2+2}$

- Re-interpretation of (*):
  in discrete: $F\,dT + \overline{F}\,d\mathcal{O}$ – closed
  ↓ "small mesh size limit"
  in continuum: $\partial_{\bar{\zeta}}\psi = m\overline{\psi}$

- ζ – conformal parametrization of a surface in $\mathbb{R}^{2+1(2)}$
- m – mean curvature ($\times$ metric)
- minimal surfaces ⤳ conformal invariance

WHAT IF THE LIMIT SURFACE $\mathcal{S}'$ IS NOT MINIMAL ?

↑ (subseq.)

WARNING :

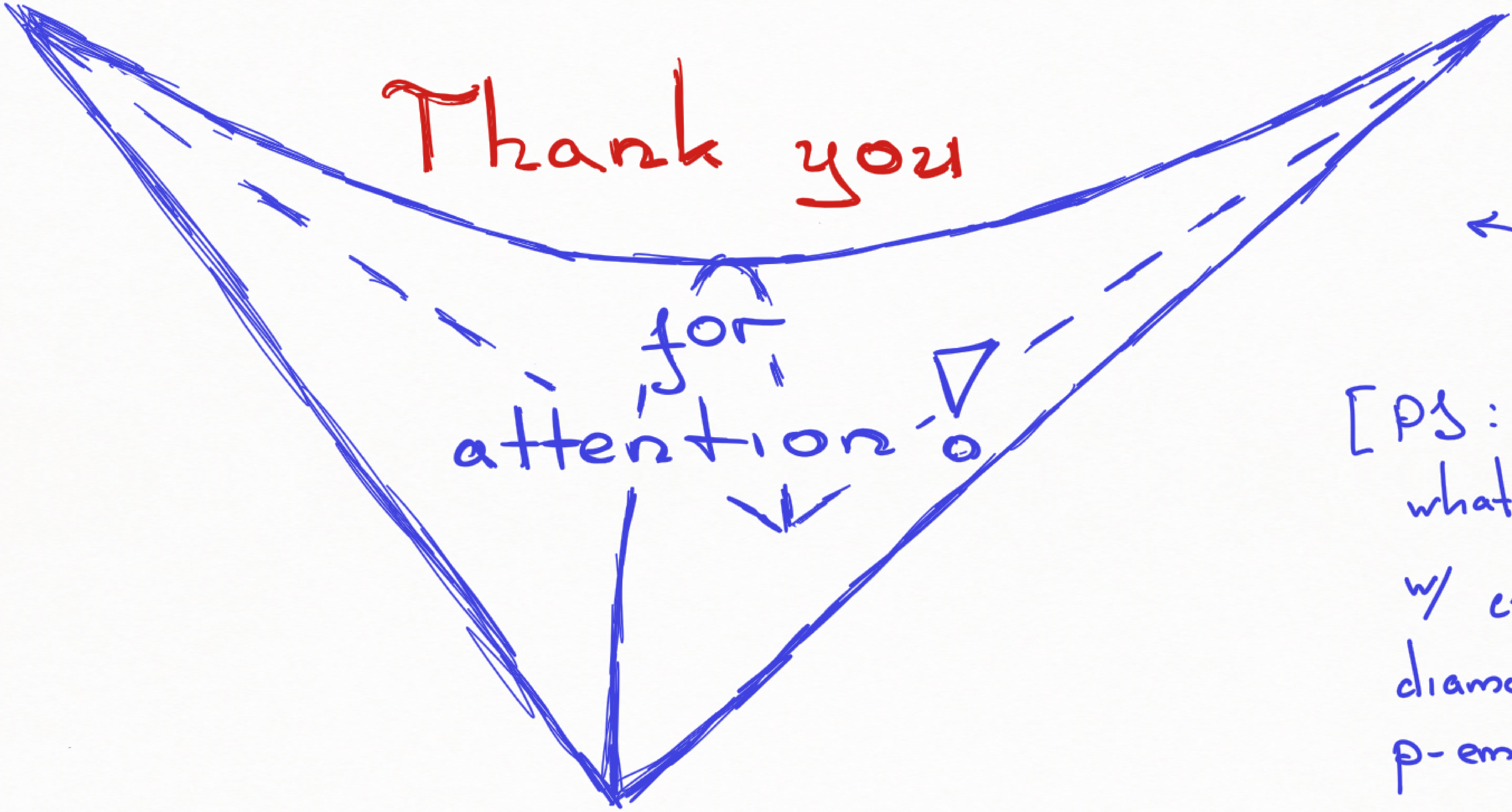
[?] It seems that there is no good reason to expect that the fluctuations are Gaussian :

the bosonization of massive free fermions ($\bar{\partial} f = m \bar{f}$) is not a free theory [cf. Coleman's correspondence]

Not developed yet [w/ Y. Wan]

However, in a liquid region of a periodic dimer model, we know/believe [Kenyon-Okounkov] that Gaussian fluctuations do appear.

! Could it be that each periodic dimer model gives rise to a "nice" definition of discrete minimal surfaces in $\mathbb{R}^{2+2}$?



["illustration" joint w/ Ramassamy]

[PS: this is what happens w/ classical Aztec diamonds under p-embeddings]