Elliptic dimer models
and
genus 1 Harnack curves

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joint work with

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Outline of the talk

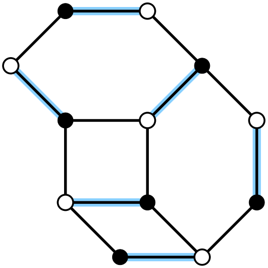
1. Generalities on the dimer model
2. Isoradial and minimal graphs
3. Fock weights
4. Elliptic dimer models on minimal graphs

1. Dimer model: definition

- $G = (V, E)$ bipartite graph, $V = B \cup W$
- Positive edge weights $\nu = (\nu_e)_{e \in E}$
- Dimer configuration = perfect matching $M \in \mathcal{M}(G)$
- For G finite, Boltzmann measure

$$\mathbb{P}(M) = \frac{\prod_{e \in M} \nu_e}{Z(G, \nu)}$$

with $Z(G, \nu) = \sum_{M \in \mathcal{M}(G)} \prod_{e \in M} \nu_e$ the dimer partition function



1. Dimer model: Kasteleyn matrix

- Assume G embedded in the plane
- Edge $e \in \mathrm{E} \rightsquigarrow$ phase $\omega_e \in S^1$ with the Kasteleyn condition: around each face of G with vertices $w_1, b_1, \ldots, w_n, b_n$,

$$\prod_{j=1}^{n} \frac{\omega_{w_j b_j}}{\omega_{w_{j+1} b_j}} = (-1)^{n+1}$$

$\rightsquigarrow$ Kasteleyn matrix $\mathrm{K} = (\mathrm{K}_{w,b})_{w \in \mathrm{W},\, b \in \mathrm{B}}$

$$\mathrm{K}_{w,b} = \sum_{e=(w,b) \in \mathrm{E}} \omega_e \, \nu_e$$

1. Dimer model: founding results

- Partition function [Kasteleyn'61, Kuperberg'98]:
  For G finite and planar,

$$Z(\mathsf{G},\nu) = |\det(\mathsf{K})|$$

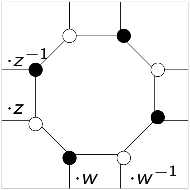
- Correlations [Kenyon'97]: For $e_1 = w_1 b_1, \cdots, e_k = w_k b_k$ distinct edges of G finite and planar,

$$\mathbb{P}(e_1,\dots,e_k \text{ dimers}) = \left(\prod_{i=1}^{k} \mathsf{K}_{w_i,b_i}\right) \det_{1\le i,j\le k}\left[\mathsf{K}^{-1}_{b_i,w_j}\right]$$

1. Dimer model: periodic case

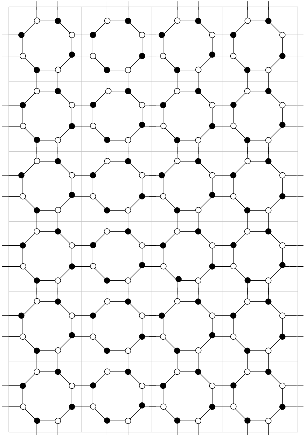
Assume G planar and $\mathbb{Z}^2$-periodic.

- K_1 a Kasteleyn matrix for $\mathrm{G}_1 = \mathrm{G}/\mathbb{Z}^2$



- Multiply edge-weights $\rightsquigarrow \mathrm{K}_1(z,w)$
- The characteristic polynomial

$$P(z,w) = \det \mathrm{K}_1(z,w)$$



"All large scale properties of the dimer model depend only on $P(z,w)$."

Kenyon-Okounkov-Sheffield'06

1. Dimer model: periodic case [KOS'06]

- The Newton polygon

$$N(G) = \text{convex hull } \{(j,k) \in \mathbb{Z}^2 \mid z^j w^k \text{ monomial in } P(z,w)\}$$

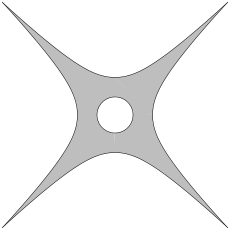
Example: $P(z,w) = 5 - z - \frac{1}{z} - w - \frac{1}{w}$

- The spectral curve

$$\mathcal{C} = \{(z,w) \in (\mathbb{C}^*)^2 \mid P(z,w) = 0\}$$

- The amoeba of $\mathcal{C}$: its image $\mathcal{A} \subset \mathbb{R}^2$ via

$$(\mathbb{C}^*)^2 \to \mathbb{R}^2, \quad (z,w) \mapsto (\log|z|, \log|w|)$$

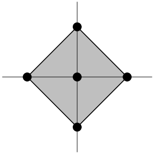


1. Dimer model: periodic case [KOS'06]

- The Newton polygon

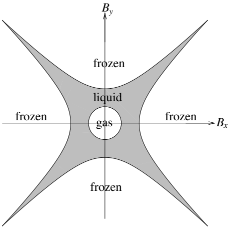
  $N(\mathrm{G}) \leftrightarrow$ (slopes of) ergodic Gibbs measures on $\mathcal{M}(\mathrm{G})$

  Example: $P(z,w) = 5 - z - \frac{1}{z} - w - \frac{1}{w}$



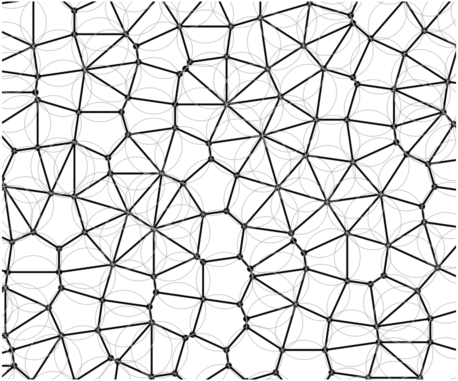
- The spectral curve $\mathcal{C} \subset (\mathbb{C}^*)^2$ is a Harnack curve

- Amoeba $\mathcal{A} \leftrightarrow$ phase diagram of the model:
  - gaseous $\leftrightarrow$ exponential decay of correlations
  - liquid $\leftrightarrow$ polynomial decay of correlations
  - frozen $\leftrightarrow$ no decay of correlations



2. Isoradial graphs

A planar graph is isoradial if all the faces are inscribed in a circle of radius 1, with center in the interior of the face.



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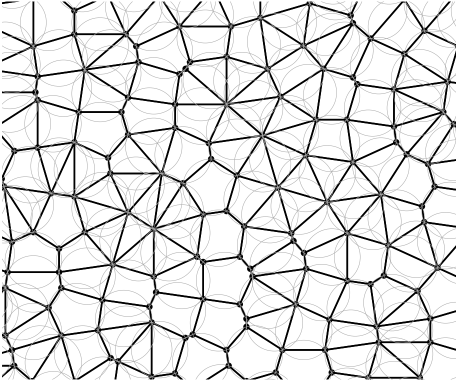
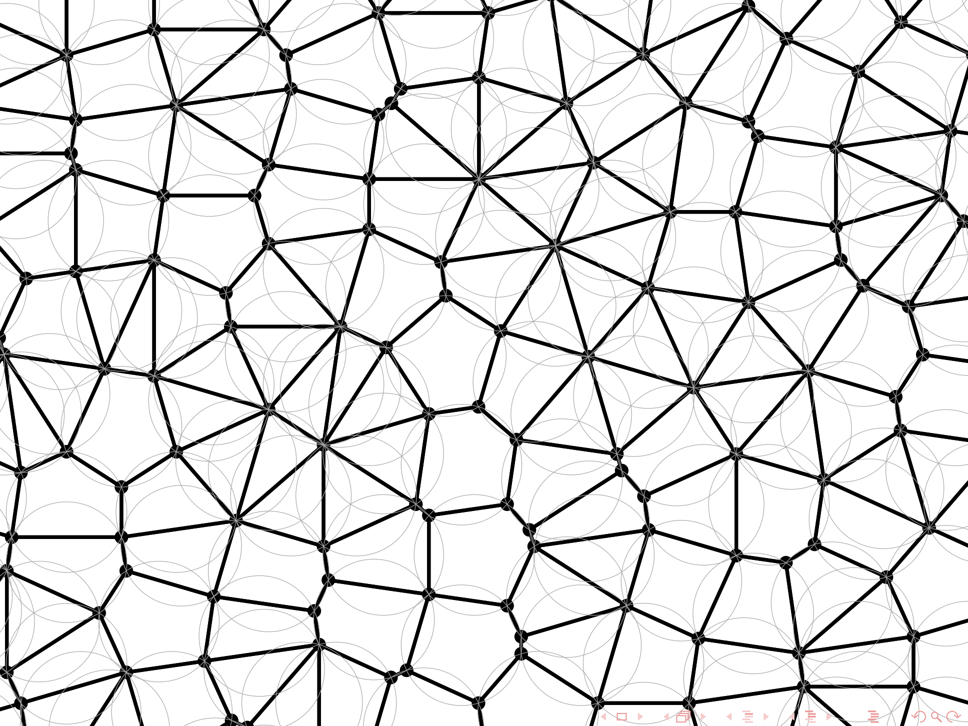
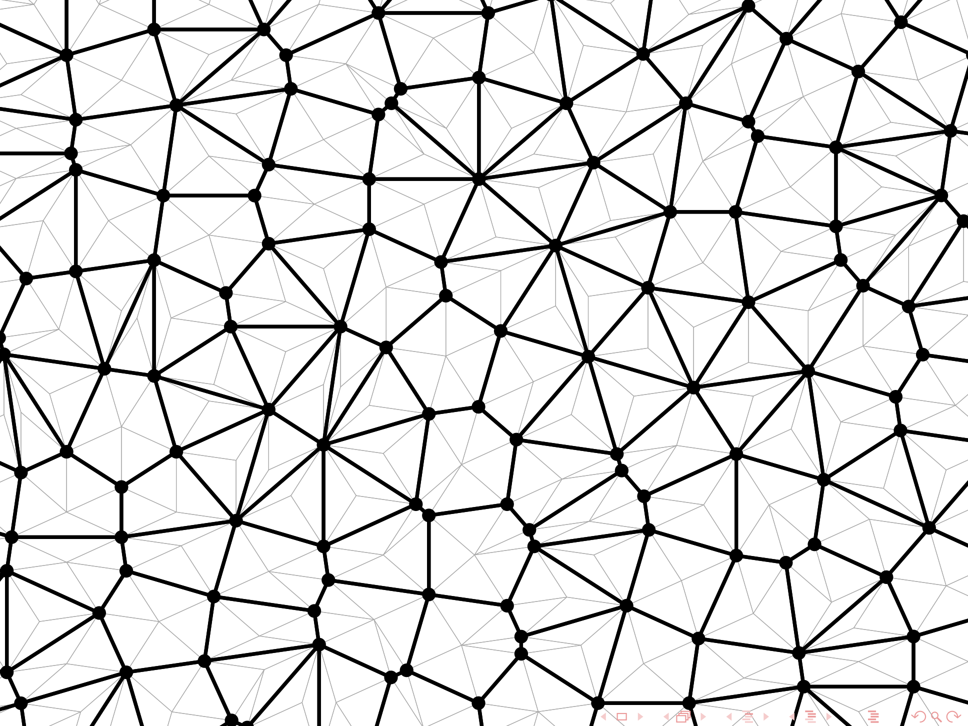
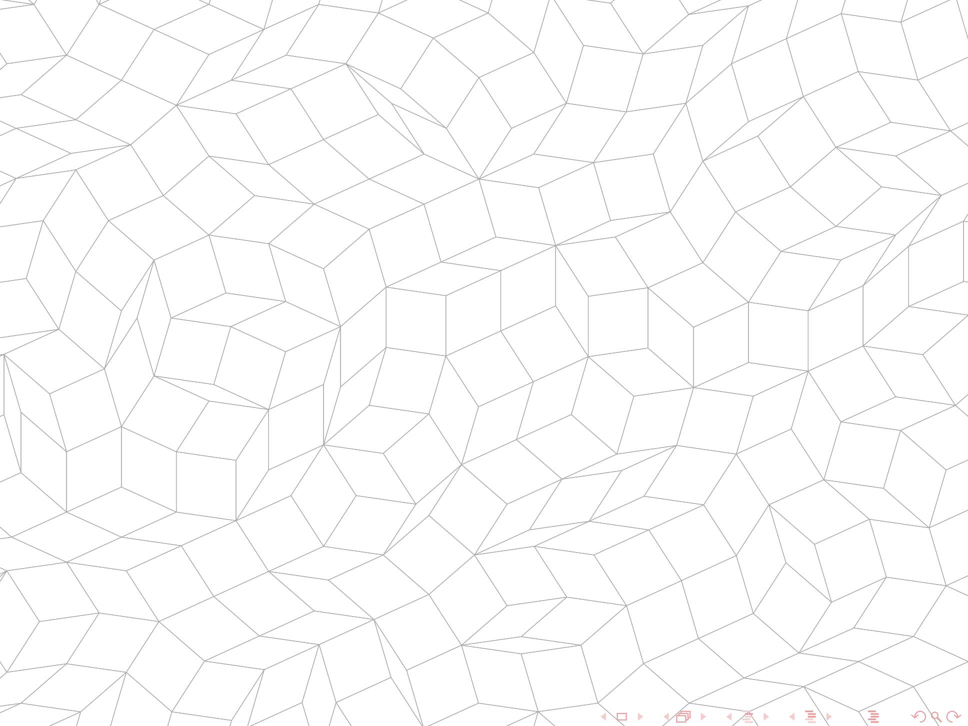
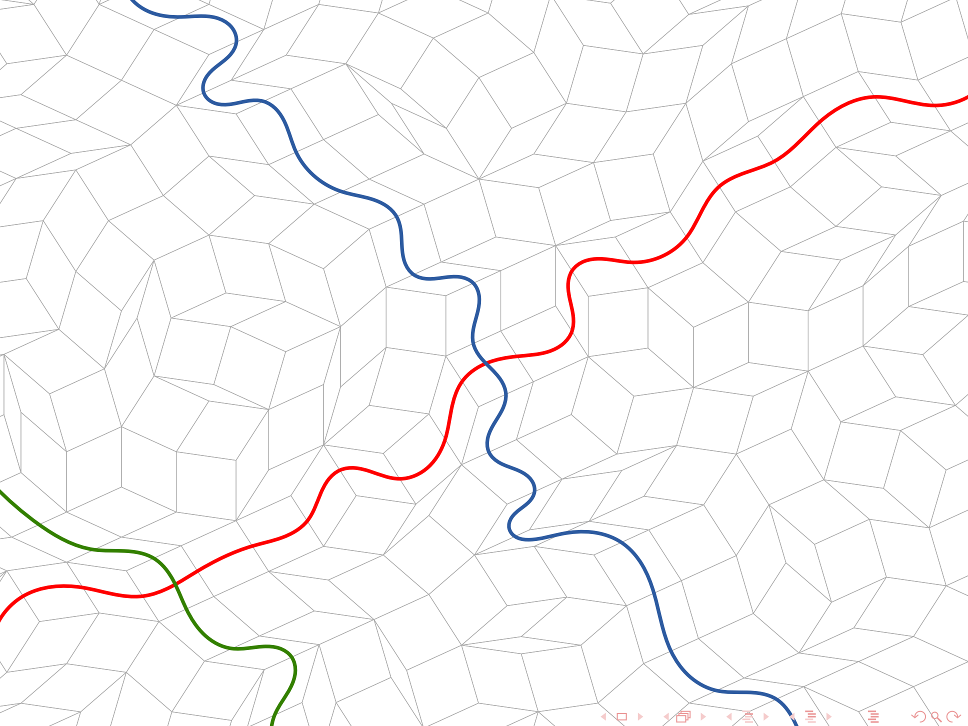


- Which planar graphs can be isoradially embedded?
- What is the space of isoradial embeddings of a given graph?



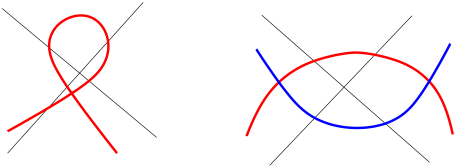






2. Isoradial graphs: Kenyon-Schlenker'05

- An infinite planar graph can be isoradially embedded ⇔



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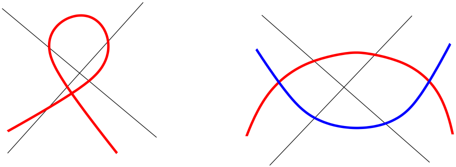
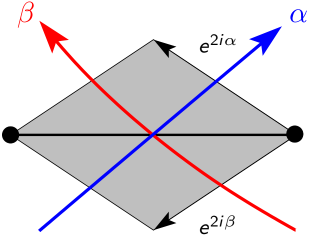
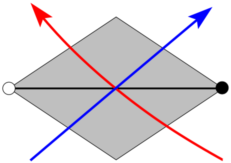


- The space of isoradial embeddings of G is an explicit set of angle maps {oriented train-tracks of G} $\longrightarrow \mathbb{R}/\pi\mathbb{Z}$



2. Minimal graphs: definition

- G bipartite $\Rightarrow$ train-tracks can be consistently oriented
  (say, white vertices on the left, black on the right)



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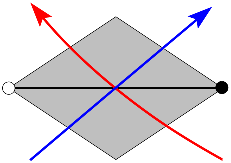
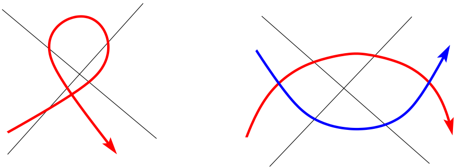


- A bipartite, planar graph G is called minimal if

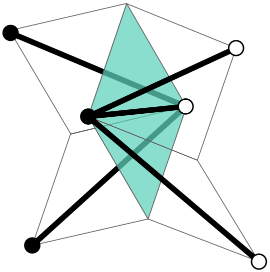


[Thurston'04, Goncharov-Kenyon'13]

2. Minimal graphs and minimal immersions

A minimal immersion of a planar graph G is an immersion of $G^\diamond$ so that:

- the faces are (possibly folded) rhombi
- rhombus angles sum up to 2π at each vertex of $G^\diamond$.



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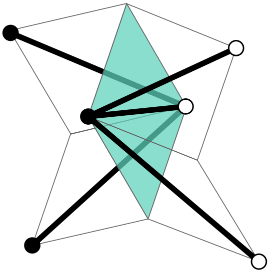


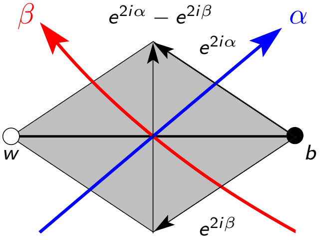
Theorem (Boutillier-C-de Tilière'19)

- *An infinite bipartite planar graph admits a minimal immersion if and only if it is minimal.*
- *The space of minimal immersions of G is an explicit set X_G of angle maps on oriented train-tracks.*

2. Minimal graphs and the Kasteleyn condition

Given any planar bipartite graph G and any angle map on oriented train-tracks, set

$$\omega_{wb} := \frac{e^{2i\alpha} - e^{2i\beta}}{|e^{2i\alpha} - e^{2i\beta}|}$$



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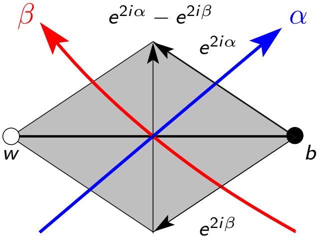

Theorem (Boutillier-C-de Tilière'19)

- *G admits an angle map such that ω satisfies the Kasteleyn condition $\Leftrightarrow$ G is minimal.*
- *If G is minimal, any angle map in X_{G} works.*

3. Fock weights: context

Theorem (Goncharov-Kenyon'13)

For any Newton polygon N, there exists a minimal graph G such that $N(\mathsf{G}) = N$.

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For G minimal, there is a bijection

$$\{\textit{weights on } \mathsf{G}\}/\sim \;\leftrightarrow\; \{\textit{Harnack curves (for } N(\mathsf{G})\textit{), one point on each oval}\}/\sim$$

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Theorem (Fock'15)

Explicit map $\leftarrow$ for all smooth algebraic curves (no check of positivity).

We will focus on the case of genus 1 !

3. Fock weights: (first) Jacobi theta function

Fix $\tau \in \mathbb{C}$ with $\Im\tau > 0$, set $q = e^{i\pi\tau}$ and

$$\theta(z) = \theta_1(z|\tau) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2} \sin\left((2n+1)z\right)$$

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- $\theta(z+\pi) = -\theta(z) = \theta(-z)$
- $\theta(z+\pi\tau) = (-qe^{2iz})^{-1}\theta(z)$
- $\theta(z) = 0 \Leftrightarrow z \in \pi\mathbb{Z} + \pi\tau\mathbb{Z}$
- θ satisfies a 3-term functional identity (Weierstrass identity)

3. Fock weights: definition

Start with an algebraic plane curve Σ of genus 1, and $t \in \Sigma$

$\rightsquigarrow \tau \in \mathbb{C}$ with $\Im\tau > 0$ so that $\Sigma \simeq \mathbb{T}(\tau) := \mathbb{C}/\pi\mathbb{Z} + \pi\tau\mathbb{Z}$

and G minimal giving the corresponding $N(\mathrm{G})$

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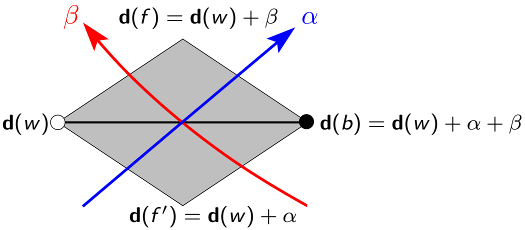
- Fix a map $\{\text{train-tracks of G}\} \longrightarrow \mathbb{R}/\pi\mathbb{Z}$
- Define a function $\mathbf{d}$ on vertices of $\mathrm{G}^{\diamond}$ by
  - set $\mathbf{d}(f) = 0$ for an arbitrary face f
  - local rule



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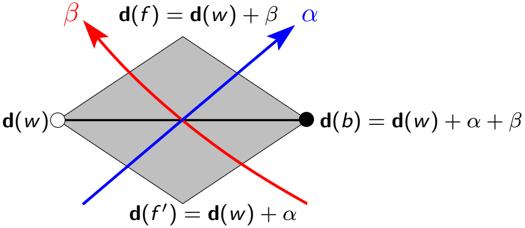


- Define $\mathrm{K} = (\mathrm{K}_{w,b})$ by

$$\mathrm{K}_{w,b} = \frac{\theta(\beta - \alpha)}{\theta(t + \mathbf{d}(f))\theta(t + \mathbf{d}(f'))}$$

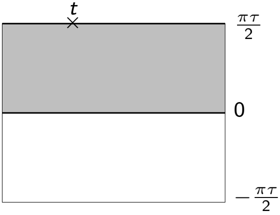
4. The elliptic Kasteleyn operator

Fix a minimal graph G and

(i) a purely imaginary τ
$\rightsquigarrow$ torus $\mathbb{T}(\tau)$ with 2 real components

(ii) t in $\mathbb{R}/\pi\mathbb{Z} + \frac{\pi\tau}{2}$

(iii) an angle map in X_G
$\rightsquigarrow$ minimal immersion of G



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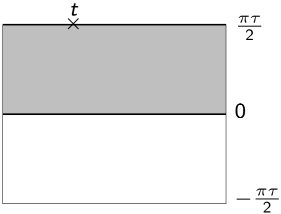


Proposition (Boutillier-C-de Tilière'20)

With these assumptions, Fock's operator K *is a Kasteleyn operator.*

Idea of proof.

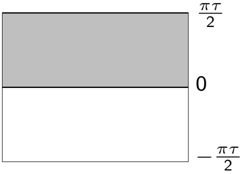
(i),(ii) + properties of θ $\Rightarrow$ the argument of $K_{w,b}$ is that of $e^{2i\alpha} - e^{2i\beta}$.
G minimal + (iii) $\Rightarrow$ this phase satisfies the Kasteleyn condition.

4. The inverse Kasteleyn operators

Theorem (Boutillier-C-de Tilière'20)

For any $u_0 \in \mathbb{R}/\pi\mathbb{Z} + [0, \frac{\pi\tau}{2}]$, *the following local formula defines an inverse* A^{u_0} *of the elliptic Kasteleyn operator* K*:*

$$A^{u_0}_{b,w} := \frac{i\theta'(0)}{2\pi} \int_{C^{u_0}_{b,w}} g_{b,w}(u)\, du$$



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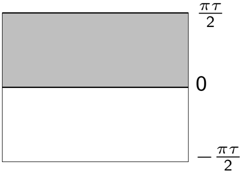
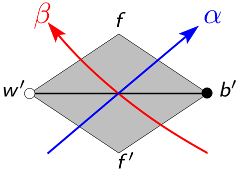


where $g_{b,w} = g_{b,x_1} g_{x_1,x_2} \cdots g_{x_n,w}$ for $b, x_1, x_2, \ldots, x_n, w$ path in $\mathsf{G}^\diamond$

$$g_{f,w'}(u) = \frac{\theta(u+t+\mathbf{d}(w'))}{\theta(u-\beta)} = g_{w',f}(u)^{-1}$$

$$g_{b',f}(u) = \frac{\theta(u-t-\mathbf{d}(b'))}{\theta(u-\alpha)} = g_{f,b'}(u)^{-1}$$



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$$\mathbb{P}^{u_0}(e_1, \ldots, e_k \text{ dimers}) = \left(\prod_{i=1}^{k} \mathsf{K}_{w_i, b_i}\right) \det_{1 \le i,j \le k} \left[A^{u_0}_{b_i, w_j}\right]$$

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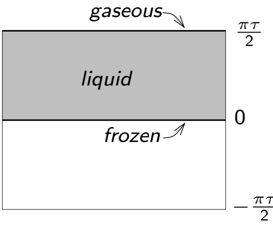
Moreover, we have the phase diagram:

- $u_0 \in \mathbb{R}/\pi\mathbb{Z} + \frac{\pi\tau}{2} \Leftrightarrow$ gaseous *(exponential decay)*
- $u_0 \in \mathbb{R}/\pi\mathbb{Z} + (0, \frac{\pi\tau}{2}) \Leftrightarrow$ liquid *(polynomial decay)*
- $u_0 \in \mathbb{R}/\pi\mathbb{Z} \Leftrightarrow$ frozen *(no decay of correlations)*



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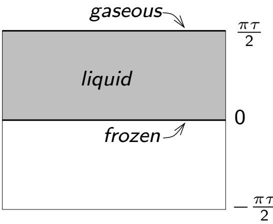


(For G periodic, it gives a local formula for the ergodic Gibbs measures of [KOS'06], and an alternative phase diagram.)

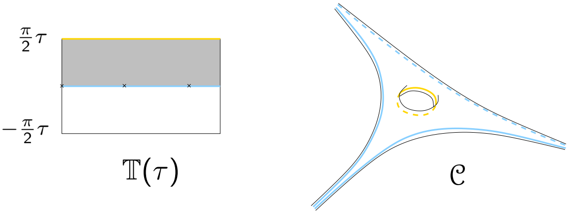
4. The periodic case

Theorem (Boutillier-C-de Tilière'20)

For any Harnack curve $\mathcal{C}$ of genus 1, $\rightsquigarrow \tau \in i\mathbb{R}_+^*$ *and* $N = N(\mathcal{C})$

there is a minimal graph G *with* $N(\mathsf{G}) = N$

and a periodic angle map $\alpha \in X_{\mathsf{G}}$ $\rightsquigarrow$ *integer point* $\in$ *int*(N)



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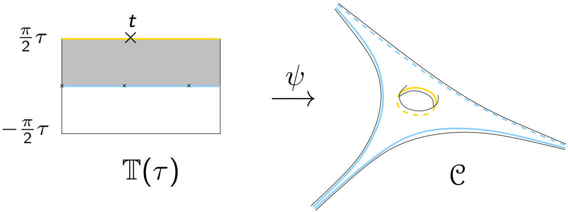
so that for any $t \in \mathbb{R}/\pi\mathbb{Z} + \frac{\tau\pi}{2}$,

the spectral curve of the dimer model *explicit parametrization*

for $(\tau, \mathrm{G}, \alpha, t)$ *is* $\mathcal{C}$. $\psi\colon \mathbb{T}(\tau) \to \mathcal{C}$



4. Connection to previous work

- For $q = e^{i\pi\tau} \to 0$, elliptic weights $\to$ critical weights of [Kenyon'02]
  isoradial $\rightsquigarrow$ minimal graphs ($\Rightarrow$ all rational Harnack curves)
  one inverse $\rightsquigarrow$ many inverses ($\Rightarrow$ all ergodic Gibbs measures)

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- G isoradial $\Rightarrow$ double $\mathsf{G} = G^D$ isoradial (hence minimal)
  $\rightsquigarrow$ Z-Dirac operator of [de Tilière'17]
  $\rightsquigarrow$ massive Laplacian operator of [Boutillier-Raschel-de Tilière'17]

- G isoradial $\Rightarrow$ Dubédat graph $\mathsf{G} = G^Q$ minimal
  $\rightsquigarrow$ Z-invariant Ising model of [Boutillier-Raschel-de Tilière'18]

Perspectives

- Show that Condition $(*)$ always holds
- Extend to arbitrary genus, where Fock weights are defined

Thank you for your attention!