1) Motivation:

τ-function is a gen. for solutions of int. systems

KP – τ-function

$\tau(t_1, t_2, \dots)$ function on $\mathbb{N}$ variables

KP eq $\quad \partial_t(\partial_x u + \partial_{xxx} u + 6u\partial_x u) = \partial_{yy}$

$$u(t,x,y) = \frac{\partial^2}{\partial t_1^2} \ln \tau(t_1, t_2, t_3 \dots)\Big|_{\substack{t_1 = t \\ t_2 = x \\ t_3 = y}}$$

depends on
Riemann surface
points on it
coordinates around

SG equation $\quad \frac{\partial^2 u}{\partial x \partial t} = \sin u$

$\tau_\pm(t_1, t_2, \dots, t_1', t_2', \dots)$

$$u = \ln \frac{\tau_+}{\tau_-}\Big|_{\substack{t_1 = t \\ t_1' = x}}$$

II Aim Explain why all these functions are specialization of one function

III Σ – Riemann surface

$\mathcal{H}$ – quasimeromorphic (QM) function on Σ.

Def QM function is a function locally $f(z) = r e^{\Phi}$ $\quad r, \Phi$ meromorphic

$$f(z) = z^k \exp \sum_{i=N}^{+\infty} z^i t^{-i}$$

Obs $\mathcal{H}$ is a group. (multiplication)

Def $p \in \Sigma$, $\quad \mathcal{H}_p$ – germs of QM function at p

Cocycle on $\mathcal{H}_p$ $f = z^k e^{\varphi}$, $g = z^l e^{\psi}$

$$c_p(f,g) = \exp 2\pi i \,\underset{p}{\mathrm{Res}} \left(\varphi\, d\psi + \frac{dz}{z}(k\psi - l\varphi)\right)$$

$$c_p(f_1 f_2, g) = c_p(f_1,g)\, c_p(f_2, g)$$

$$c_p(f,g) = c_p(g,f)^{-1}$$

Def $\hat{\mathcal{H}}_p \overset{\text{set}}{=} \mathcal{H} \times \mathbb{C}^*$ $(f,x)(g,y) = (fg,\, xy\, c(f,g))$

$\mathcal{H}_p^+ = \{(f,x) \mid f$ is nonsingular $\neq 0\}$

↑ commutative since $c(f,g)=0$ for f,g holomorphic $\neq 0$

$z^k e^{\varphi}$ is holo $\Longleftrightarrow$ $k=0$, φ is holo

$\mathcal{H}^- = \hat{\mathcal{H}}/\mathcal{H}^+ = \{z^k e^{\sum_{k>0} t_k z^{-k}}\}$

↑ depends on the choice of z

$\mathcal{H}_p^+$ has a canonical representation

$I : (f,x) \to x$

$\hat{\mathcal{H}}_p$ has a canonical representation

$\mathcal{B}_p = \mathrm{Ind}_{\mathcal{H}_p^+}^{\hat{\mathcal{H}}_p} I$; $\mathcal{B}_p \overset{\text{vect}}{=} \{$function V on $\hat{\mathcal{H}}_p$ such that $V(fx) = I(f)V(x)$, $f \in \mathcal{H}^+$

$\mathcal{B}_p =$ functions on $\hat{\mathcal{H}}_p/\mathcal{H}_p^+ =$

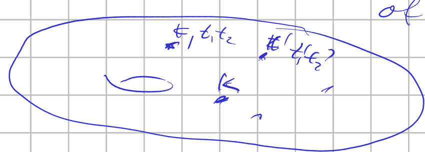
$=$ functions of $(k, t_1, t_2, \dots)$

IV Globalization

$$\hat{\mathcal{H}}(\Sigma) = \widehat{\left(\bigotimes_{p\in\Sigma} \mathcal{H}_p\right)}$$

$\mathcal{B}(\Sigma) = \bigotimes_p \mathcal{B}_p$ "=" functions on collections of points and parameters $k, t_1, t_2 \dots$ at each point

generalized divisors ; →



Thm $\mathcal{H} \subset \hat{\mathcal{H}}(\Sigma) \Longleftrightarrow \forall f,g$ – global QM functions

$$c(f,g) := \prod_p c_p(f,g) = 1$$

(generalized) Weil reciprocity

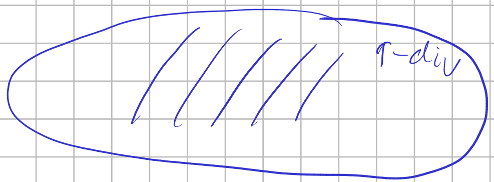
Def τ is an element in $\hat{\mathcal{L}}(\Sigma)$
invariant under $\mathcal{L}$ $(\forall f \quad f \circ \tau = \tau)$

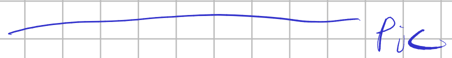
Quasi-divisor = collection of points
+ ~~multiplicities~~ k + parameters $t_1\ t_2, \ldots$
at each point

$$f \to (f) \qquad \frac{\text{qusi-divisons}}{\text{principal q-divison}} = \mathrm{Pic}(\Sigma)$$

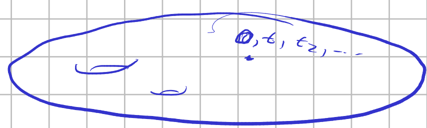
τ : quasi divisors $\to \mathbb{C}$

invariance of τ $\iff$ $\tau(d + (f)) = \tau(d)\, \rho(d, f)$





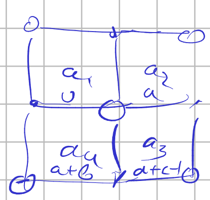
KP $\to$ Σ, p



SG $\to$ Σ - hyper elliptic, p_1, p_1' - ramification points.

$$\tau_+ = \tau(+1, t_1 t_2 \ldots, -1\ t_1' t_2 \ldots)$$
$$\tau_- = \tau(-1\ t_1 t_2, \ t_1', t_2 \ldots)$$

GK int. system.



$\supseteq \mathcal{D}$

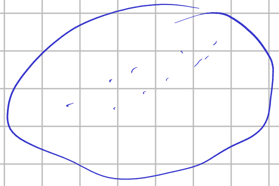
$$x_1 = \frac{a_2^2}{a_4^2} \ldots$$

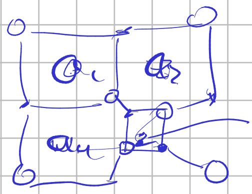
a - coordinates with divisors of the spectral curve with support at infinity

if $d = \sum n_i [z_i]$

$$a_d = \tau(n_i[z_i]) = \qquad n_i n_j$$

$$= \Theta(n_i z_i) \prod_{i<j} \Theta(z_j - z_i)$$





$$\frac{a_2^2 + a_4^2}{a_3}$$

$\tau : (q\,d\omega) \to 0$

$$x_j = \prod_j a_j^{\varepsilon_{ij}}$$

ε_{ij}