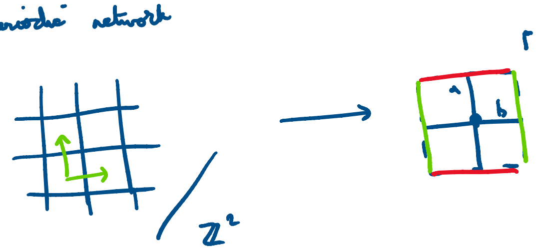
INVERSE SPECTRAL PROBLEM FOR BIPERIODIC NETWORKS

Friday, November 20, 2020 10:02 AM

Γ biperiodic network



$a, b \in \mathbb{C}^*$

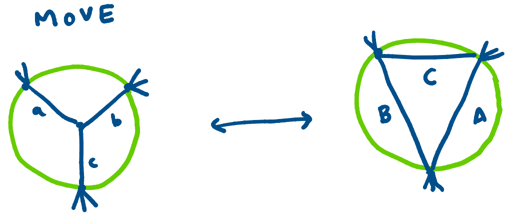
$\lambda a, \lambda b$

$\left(c : E(\Gamma) \to \mathbb{C}^*\right) / \mathbb{C}^*$ conductance

$c = \lambda c, \ \lambda \in \mathbb{C}^*$

$$\mathcal{R}_\Gamma := \{ c : E(\Gamma) \to \mathbb{C}^* \} / \mathbb{C}^* \cong (\mathbb{C}^*)^{E(\Gamma) - 1}$$

Y – Δ MOVE

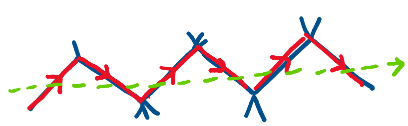


$$\boxed{A = \frac{bc}{a+b+c} \quad \vdots}$$

$$\Gamma \xrightarrow{Y-\Delta} \Gamma'$$

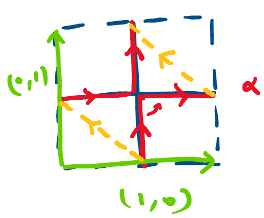
$$\mathcal{R}_\Gamma \overset{\mu_{Y-\Delta}}{\longleftarrow - - \rightarrow} \mathcal{R}_{\Gamma'}$$

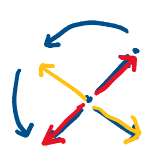
ZIG-ZAG PATH (strands, train-tracks, trips ...)

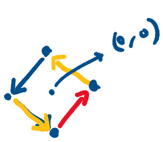


$H_1(\mathbb{T}, \mathbb{Z}) \cong \mathbb{Z}^2$

$\alpha \longmapsto (1,1)$







Newton polygon of Γ

$N(\Gamma)$

1) Closed convex integral polygon

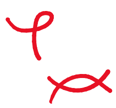
$$\sum_\alpha [\alpha] = 0$$

2) Centered at (0,0) and symmetric under rotation by π.

TOPOLOGICAL EQUIV

$\Gamma \sim_t \Gamma'$ if $\exists$ a sequence of Y-Δ moves taking $\Gamma \longrightarrow \Gamma'$.

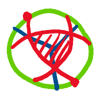
Proposition : [Goncharov-Kenyon '13]

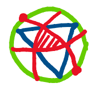


$\{\underline{\text{Minimal}} \text{ networks}\} / \sim_t \xrightarrow{\sim} \{\text{Polygons satisfying } 1), 2)\}$

Ψ

N





ELECTRICAL EQUIV.

Fix N. $(\Gamma, c) \sim_e (\Gamma', c')$ if $\exists$ a sequence of Y-Δ taking $(\Gamma, c) \longrightarrow (\Gamma', c')$.

$$\mathcal{R}_N := \{(\Gamma, c) : N(\Gamma) = N\} / \sim_e$$

resistor network cluster variety

affine charts — $\mathcal{R}_\Gamma \overset{\mu_{Y-\Delta}}{\longleftarrow - - \rightarrow} \mathcal{R}_{\Gamma'}$

$\cap$

$\mathcal{R}_N :=$ glue all the $\mathcal{R}_\Gamma$ using the Y-Δ moves.

MOTIVATION

1) Classification of networks.

2) Action - angle coordinates for an "unknown" integrable system on $\mathcal{R}_N$.

$$\Gamma \xrightarrow[\text{Y-}\Delta \text{ moves}]{\text{sequence of}} \Gamma$$

$$\begin{array}{ccc} \mathcal{R}_\Gamma & \overset{\mu_{Y-\Delta}}{\longleftarrow - - - \rightarrow} & \mathcal{R}_\Gamma \\ \cap & & \cap \\ \mathcal{R}_N & \overset{\mu_{Y-\Delta}}{\longleftarrow - - \rightarrow} & \mathcal{R}_N \end{array}$$

<u>discrete dynamics</u>

3) David's talk on work with Cedric and Beatrice.

LINE BUNDLE LAPLACIAN [Kenyon 2011]

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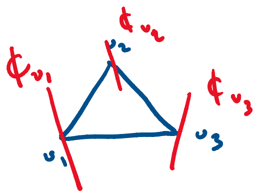
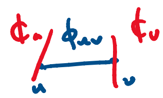


$$\Delta : \mathbb{C}^{V(\Gamma)} \longrightarrow \mathbb{C}^{V(\Gamma)}$$
$$f \longmapsto (\Delta f)$$

$$(\Delta f)(u) = \sum_{v \to u} c(uv)\big[\underset{\mathbb{C}_u}{f(u)} - \phi_{vu} \underset{\mathbb{C}_v}{f(v)}\big]$$

$\phi_{uv} : \mathbb{C}_u \longrightarrow \mathbb{C}_v$

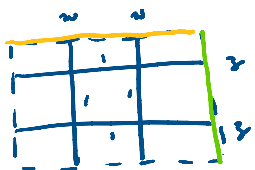


$\in \mathbb{C}_u$

$\phi_{vu} = \phi_{uv}^{-1}$

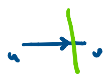
$\phi_{uv} \in \mathbb{C}^*$

ϕ_{uv}



For all faces f in Γ,

$\prod_f \phi_{uv} = 1$ "flat connections"



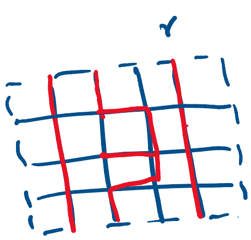
$\phi_{uv} = z \in \mathbb{C}^*$

$\phi_{uv} = w \in \mathbb{C}^*$

Theorem [Forman 1993]

$$\det \Delta(z,w) = \sum_{\substack{\text{CRSFs} \\ \gamma}} \Big(\prod_{e \in \gamma} c(e)\Big) \Big(\prod_{\substack{\text{cycles} \\ \eta \text{ in } \gamma}} \big(2 - z^i w^j - z^{-i} w^{-j}\big)\Big)$$

$[\eta] = (i,j)$



$$\prod_{e \in \gamma} c(e) \quad \Big(2 - \frac{1}{w} - w\Big)^3$$

SPECTRAL TRANSFORM

Definition/theorem [G'19]

$$\kappa_{\Gamma, v} : \mathcal{X}_N \dashrightarrow \{(C, S, \nu)\}$$

v vertex in Γ

I) C is spectral curve.
$= \{(z,w) : \det \Delta(z,w) = 0\} \subseteq (\mathbb{C}^*)^2.$

1) $(1,1) \in C$ and its a node.

2) $(z,w) \overset{\sigma}{\longleftrightarrow} \left(\frac{1}{z}, \frac{1}{w}\right).$

3) Newton polygon of $C = N(\Gamma)$.

3) Newton polygon of C = $N(\Gamma)$.

II) $S = \sum_{i=1}^{g} (p_i, q_i) \qquad (p_i, q_i) \in C$

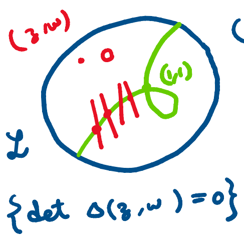
g = # interior points of $N(\Gamma)$ $\underline{\underline{-1}}$

= genus of a generic C.

$(z,w) \in (\mathbb{C}^*)^2 \qquad v \to \begin{pmatrix} \circ \\ \vdots \\ \circ \end{pmatrix} \longmapsto s_v$ section of $\mathcal{L}$

$\mathcal{L} = \operatorname{coker} \Delta(z,w)$

$\mathbb{C}^{\Gamma(v)} \xrightarrow{\Delta(z,w)} \mathbb{C}^{\Gamma(v)} \longrightarrow \operatorname{coker} \Delta(z,w) \longrightarrow 0$

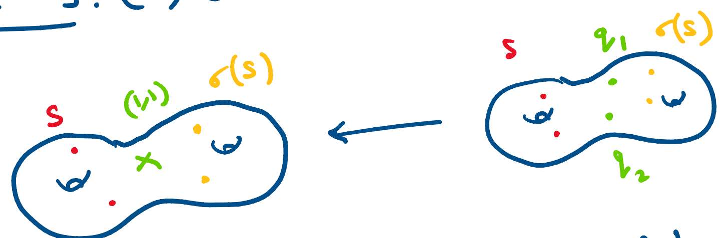


$\mathcal{L}$ is a line bundle on C.

$S = \underline{\underline{\mathrm{div}}}(s_v) = \{(p,q) : s_v(p,q) = 0\}$

$(z,w) \xleftrightarrow{\sigma} \left(\frac{1}{z}, \frac{1}{w}\right)$

<u>Theorem [G'19]</u>: $(*)\ S + \sigma(S) - q_1 - q_2 \sim K_{\hat{C}}$ — canonical divisor

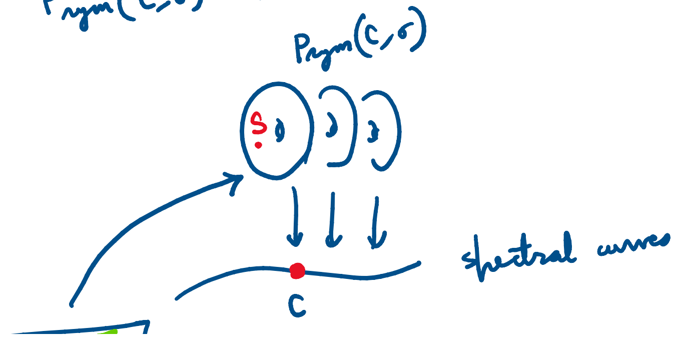


$\exists\ \partial$ ~ 1-form on $\hat{C}$ that vanishes at $S, \sigma(S)$, and has simple poles at q_1, q_2.

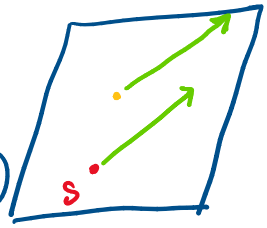
$$\begin{array}{ccccc} \{\deg g \text{ effective divisors}\} & \dashrightarrow & \mathrm{Jac}(C) & & \mathbb{C}^g/\Lambda \\ \uparrow & & \uparrow & & \uparrow \\ (*) & \dashrightarrow & \mathrm{Prym}(C,\sigma) & & \mathbb{C}^{g/2}/\Pi \end{array}$$

Theorem [G'19] 1) $\mathcal{X}_N \xleftarrow[\sim]{K_{\Gamma,v}} \{(C, S, v)\}$

2) $\mathcal{X}_N \longrightarrow \mathcal{X}_N$



2)
$$\begin{tikzcd} \mathcal{R}_N \arrow[r] \arrow[d, dashed, "K_{\sigma,\nu}"'] & \mathcal{R}_N \arrow[d, dashed, "K_{\tau_\nu}"] \\ (C,S,\nu) \arrow[r, mapsto] & (C,S',\nu') \end{tikzcd}$$



dimers

$\mathcal{R}_N \hookrightarrow (\mathcal{X}_N, \{\cdot,\cdot\})$

$\{\cdot,\cdot\}\big|_{\mathcal{R}_N} = 0$

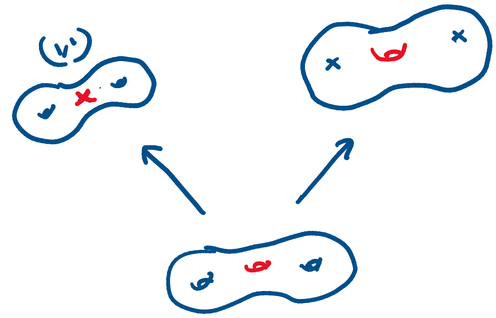
I) What is the Poisson structure on $\mathcal{R}_N$?

II) Spectral transform for Ising?

Kasteleyn   Laplacian   (?)

Jacobian   Prym   ?

III) Massive Laplacian?



IV) Spider move   Y-Δ move   Ising Y-Δ

Fay's trisecant identity   Fay's quadrisecant identity   ?