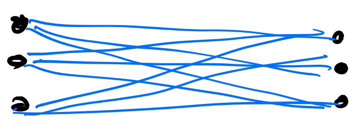
The Multidimer model (joint w/ Cosmin Pohoata)

G graph, $\nu : E \to \mathbb{R}_{>0}$ edge weights

$\vec{N} = \{N_v\}_{v \in V}$, $N_v \in \mathbb{N}$, multiplicities at each vertex.

$G_{\vec{N}}$ = replace each vtx with N_v vertices
replace each edge with $K_{N_v N_w}$

$G_{\vec{N}}$ 

G 

<u>Def</u> An <u>$\vec{N}$-dimer cover</u> is a dimer cover of $G_{\vec{N}}$.

Let $\Omega_{\vec{N}} = \{\vec{N}\text{-dimer covers}\}$

$m \in \Omega_{\vec{N}}$ has <u>weight</u> $\nu(m) = \prod_{e \in E} \nu_e^{m(e)}$

$$\boxed{Z(\nu, \vec{N}) = \sum_{m \in \Omega_{\vec{N}}} \nu(m)}$$ partition function.

We associate a variable x_v to vertex v.

define a polynomial

$$\boxed{P = \sum_{e=uv} \nu_e x_u x_v}$$

(sum over all edges of G)

Thm $$Z(\nu) := \exp(P) = \sum_{\vec{N} \geq 0} Z(\nu, \vec{N}) \frac{\vec{x}^{\vec{N}}}{\vec{N}!}$$

$\prod x_v^{N_v}$

$\prod N_v!$

Pf $\exp(P)$ is the exponential g.f. for labeled multisets of edges, or equivalently, multidimer covers. $\square$

Asymptotics

Choose $\{\alpha_v\}_{v \in V}$ $\alpha_v \in (0,1)$, $\sum_{v \in V} \alpha_v = 2$

Take $N_v \to \infty$ simultaneously, so that

$$\boxed{\frac{N_v}{M} \to \alpha_v}$$ where $M = \frac{1}{2} \sum_v N_v$ = total # diners.

Then $\frac{1}{M!} Z(\nu,\vec{N}) = \frac{\vec{N}!}{M!}[x^{\vec{N}}]Z(\nu)$

$$= \frac{\vec{N}!}{M!^2}\frac{1}{(2\pi i)^n}\int_C \frac{P^M}{\vec{x}^{\vec{N}}}\prod_v \frac{dx_v}{x_v}$$

For large M use saddle point method: find critical point of integrand.

defined by eqns' $\boxed{\frac{x_v P_{x_v}}{P} = \alpha_v}$

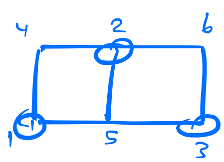
There is a unique solution $\vec{x} = \vec{x}_0$ (up to scale) by log-concavity.

$\Rightarrow$

Free energy

$$F(\nu,\vec{\alpha}) := \lim_{M\to\infty}\frac{1}{M} \log\left(\frac{Z(\nu,\vec{N})}{M!}\right)$$

$$\boxed{F(\nu,\vec{\alpha}) = \log P(\vec{x}_0) - \sum_v \alpha_v \log x_{0,v} - 2\underbrace{h\left(\frac{\vec{\alpha}}{2}\right)}_{-\sum_v \frac{\alpha_v}{2}\log\frac{\alpha_v}{2}} + \log 4}$$

Example 

$\alpha_v \equiv \frac{1}{3}$ $\nu \equiv 1$

$P = x_1x_4 + x_1x_5 + x_2x_4 + x_2x_5 + x_2x_6 + x_3x_5 + x_3x_6$

$$\begin{cases} \dfrac{x_1 x_4 + x_1 x_5}{P} = \frac{1}{3} \\ \vdots \\ \qquad = \frac{1}{3} \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} \qquad F = \log \left(\frac{11+5\sqrt{5}}{54} \right)^{\frac{1}{3}}$$

Edge fluctuations

Fix $\vec{\alpha}$. Let $\vec{x}_0$ be equilibrium vertex weights.

Let $c_e = \nu_e x_{0,u} x_{0,v}$ where $e = uv$

equilibrium edge weights, scaled to sum to 1

After placing M dimers, each edge now has a Poisson # of dimers on it, mean $\boxed{Mc_e}$

Thm: $\hat{X}_e = \left\{ \dfrac{X_e - Mc_e}{\sqrt{M}} \right\}$ converges to a multidimensional Gaussian RV with covariance

$$\mathrm{Cov}(\hat{X}_e, \hat{X}_f) = S(e,f), \quad \text{where}$$

S = "transfer cycle current" matrix

= kernel for the determinantal process for complement of a spanning tree

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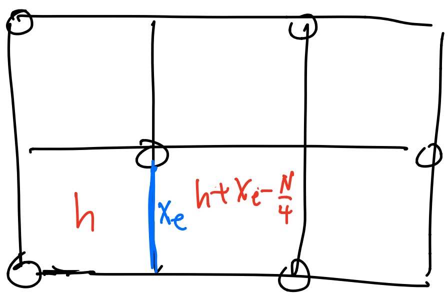
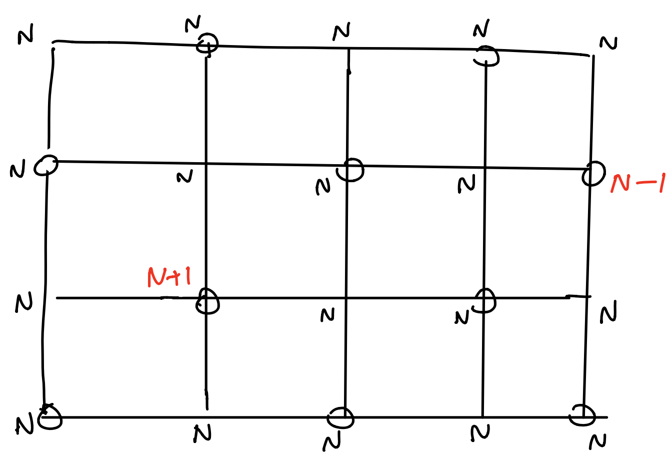
Cor

For planar graphs like $\mathbb{Z}^2$, there is a height function; its fluctuations form a Gaussian Free Field for the dual graph (with dual conductances $\frac{1}{C_e}$).



Coulomb Gas



Suppose G is δ-regular and bipartite.

Take $\nu_e \equiv 1$

Let $N_v = N + p_v$, where $\begin{cases} p_v \text{ fixed as } N \to \infty \\ \sum_w p_w = 0 = \sum_b p_b \end{cases}$

We can perturb saddle-point equations to first order in $d\alpha_v$

We find

$$\boxed{\text{Thm} \quad d^2 F = \frac{n}{4}\Big(\underbrace{-d\tilde{\alpha}^t \Delta^{-1} d\tilde{\alpha}}_{\text{interaction energy}} + \underbrace{d\tilde{\alpha}^t \cdot d\tilde{\alpha}}_{\text{internal energy}} \Big) \quad \text{where } d\tilde{\alpha}_v = \begin{cases} d\alpha_v & v \in W \\ -d\alpha_v & v \in B \end{cases}}$$

Internal energy: sum of squares of charges

Interaction energy: coulomb potential

define the charge $q_v = \begin{cases} p_v & v \in W \\ -p_v & v \in B \end{cases}$

Then $\begin{cases} \text{like charges repel} \\ \text{opposite charges attract} \end{cases}$

In $\mathbb{Z}^2$ potential $\log r \longleftrightarrow$ force $\frac{q_v q_{v'}}{r}$

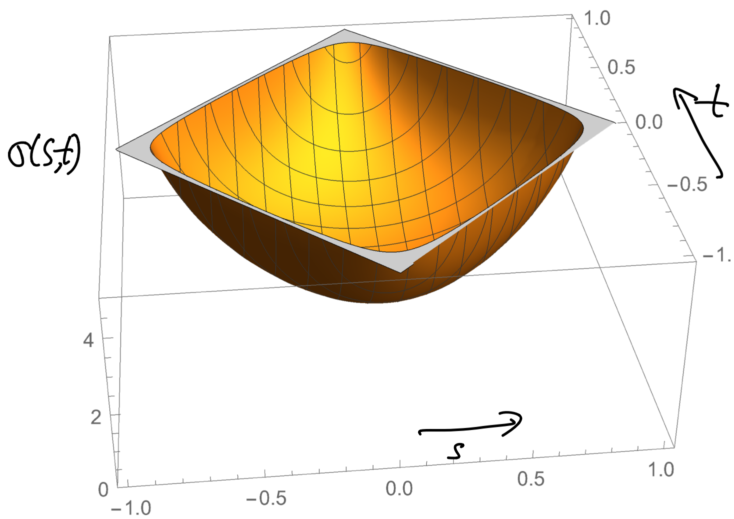
$\mathbb{Z}^3$ potential $\frac{1}{r} \longleftrightarrow$ force $\frac{q_v q_{v'}}{}$

r^2

Other applications

* Surface tension and limit shapes

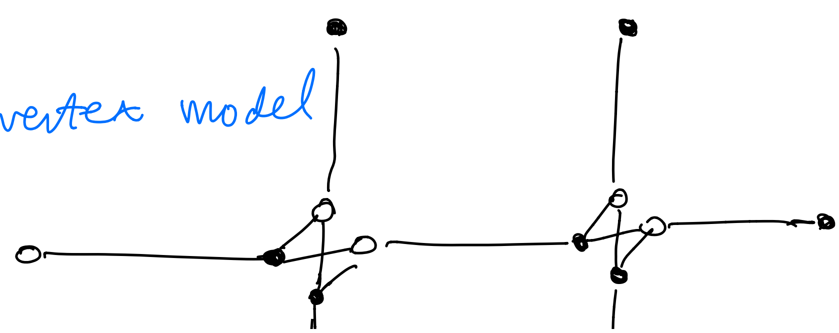
$$F(X,Y) = \log(e^{X} + e^{-X} + e^{Y} + e^{-Y}) \qquad \text{for } \mathbb{Z}^2$$



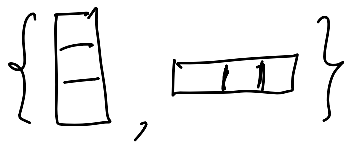
$$F(X,Y) = \log(e^{X} + e^{-X} + e^{Y} + e^{-Y} + e^{Z} + e^{-Z}) \qquad \text{for } \mathbb{Z}^3$$

...

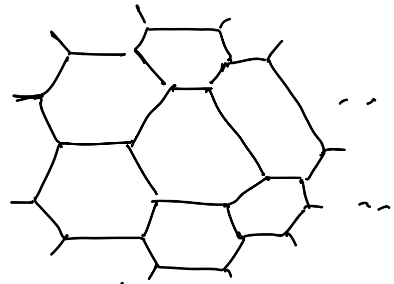
* Multi-6-vertex model



* Other random tiling models

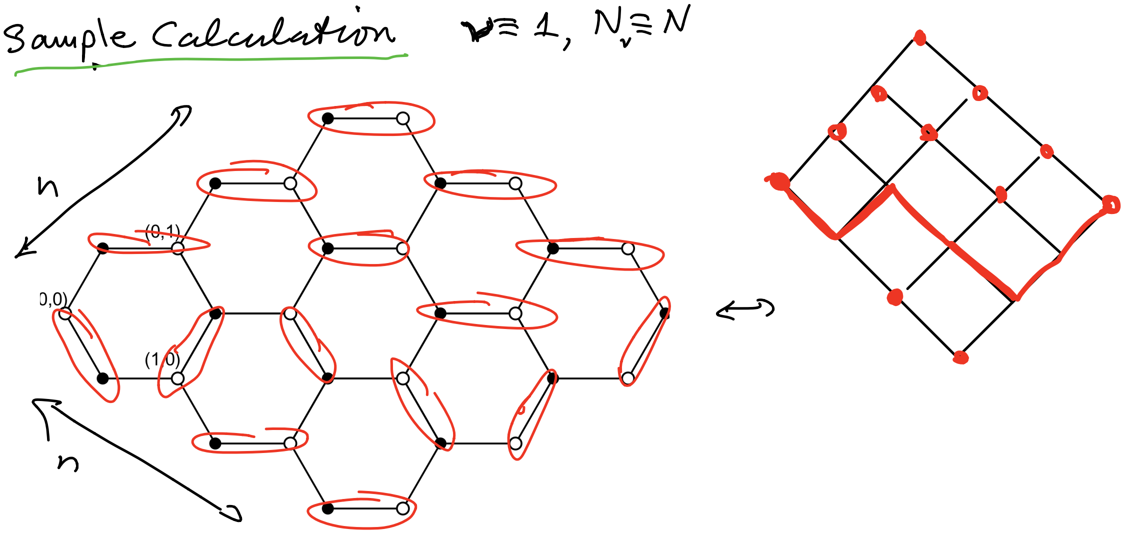


3×1, 1×3 bars



random honeycombs

Sample Calculation $\quad \nu \equiv 1, \; N_\nu \equiv N$



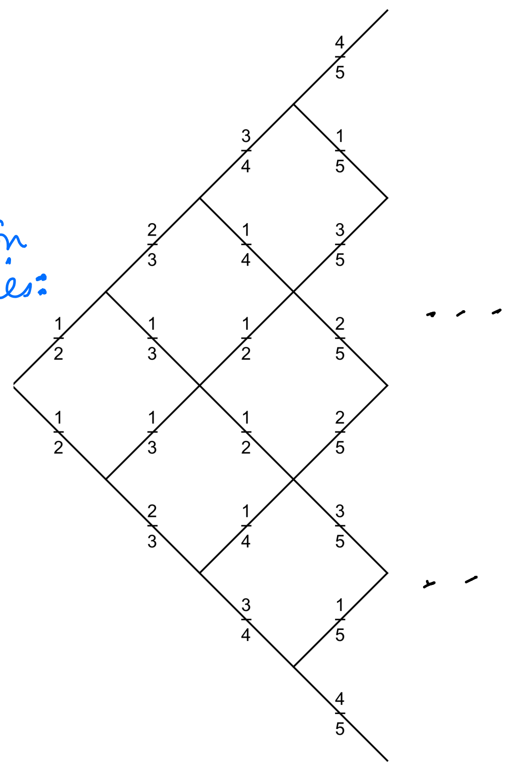
equilibrium equation

$$\boxed{\frac{x_{ij}}{x_{ij}+x_{i-1,j}+x_{i,j-1}} + \frac{x_{ij}}{x_{ij}+x_{i+1,j}+x_{i+1,j-1}} + \frac{x_{ij}}{x_{ij}+x_{i,j+1}+x_{i-1,j+1}} = 1}$$

+ boundary conditions $x_{ij} = 0$ if $i<0$, or $j<0$.

<u>In limit $n\to\infty$</u> $\boxed{x_{ij} = (i+j)!\binom{i+j}{i}}$

transition probabilities:



We get a <u>Polya Urn</u>: the "multidimen" random walk is <u>uniformly distributed</u> at time k $\forall k\geq 0$.