Planar Site Percolation and Benjamini-Schramm Conjecture

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- Bernoulli(p) percolation ($p \in [0, 1]$): the state of each vertex is an independent Bernoulli(p) random variable.
- Question: when does an infinite cluster exist?

Site Percolation: Examples

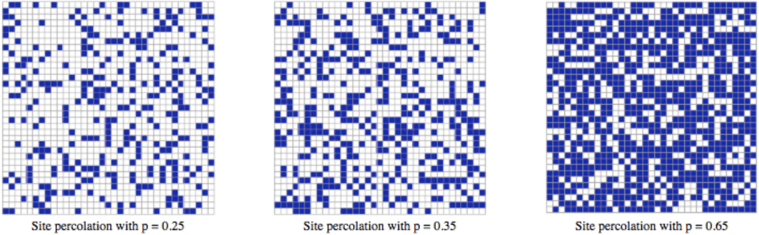

Figure: Site Percolation on the Square Grid. $p_c \approx 0.59$ (Picture by Kirkpatrick)

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Graph Structures: Planarity, Transitivity

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- G is quasi-transitive if there exists $\Gamma \subseteq \mathrm{Aut}(G)$, such that all the vertices are in finitely many different orbits under the action of Γ.

Graph Structures: Amenability, Number of Ends

- G is amenable if

$$\inf_{K \subseteq V(G), |K| < \infty} \frac{|\partial_E K|}{|K|} = 0, \tag{1}$$

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- The number of ends of a connected graph is the supremum over its finite subgraphs of the number of infinite components that remain after removing the subgraph.

B-S Conjectures on Planar Site Percolation

Conjecture

(Benjamini-Schramm 1996)Suppose G is planar, and the minimal degree in G is at least 7. Then at every p in the range $(p_c, 1 - p_c)$, there are infinitely many infinite open clusters. Moreover, we conjecture that $p_c < \frac{1}{2}$, so the above interval is nonempty.

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Conjecture

(Benjamini-Schramm 1996)Let G be a planar graph. Let $p = \frac{1}{2}$ be the probability that a vertex is open and assume that a.s. percolation occurs in the site percolation on G, then almost surely there are infinitely many infinite clusters.

History

- (Benjamini-Schramm 2000, JAMS)Let G be a transitive, nonamenable, planar graph with one end. Then $0 < p_c(G) < p_u(G) < 1$, for Bernoulli bond or site percolation on G.

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- (Z. Li 2017) Proved the two conjectures for vertex-transitive triangular tilings of the hyperbolic plane with vertex degree $d \geq 7$.
- (Halslegrave-Panagoitis 2020)Planar graphs of minimum degree at least 7 have $p_c^{site} < \frac{1}{2}$.

Main Results

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Let G be an infinite, connected, locally finite, transitive, planar graph in which each vertex has degree at least 7. Consider the i.i.d. Bernoulli(p) site percolation of G. Then

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2. *For every p in the range $(p_c^{site}, 1 - p_c^{site})$, there are infinitely many infinite open clusters and infinitely many infinite closed clusters a.s.*
3. *For every p in the range $[0, 1]$, a.s. there exists at least 1 infinite open or closed cluster.*

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1. $p_u^{site} + p_c^{site} \geq 1$.
2. *Assume each vertex is open independently with probability $\frac{1}{2}$. Assume that a.s. percolation occurs in the site percolation on G, then almost surely there are infinitely many infinite 1-clusters and infinitely many infinite 0-clusters.*

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4. G has infinitely many ends, $p_u^{site} = 1$.

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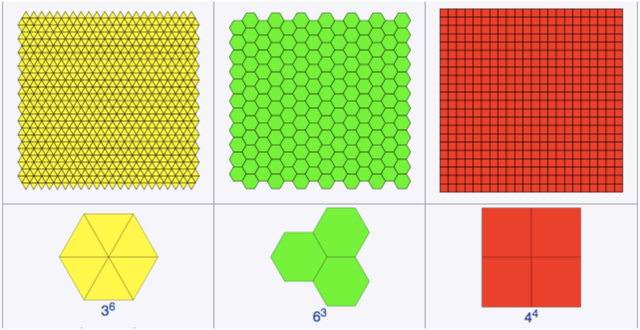
4. 11 vertex-transitive Archimedean embeddings of $\mathbb{R}^2$: [6,6,6], [3,12,12],[4,6,12],[4,8,8],[4,4,4,4],[3,6,3,6],[3,4,6,4]; [3,3,3,3,6],[3,3,4,3,4],[3,3,4,3,4], [3,3,3,3,3,3]; in each of which the Bernoulli($\frac{1}{2}$) percolation a.s. has no infinite 1-clusters (prove by symmetry).

11 vertex-transitive Archimedean Tilings of $\mathbb{R}^2$, picture from Wikipedia



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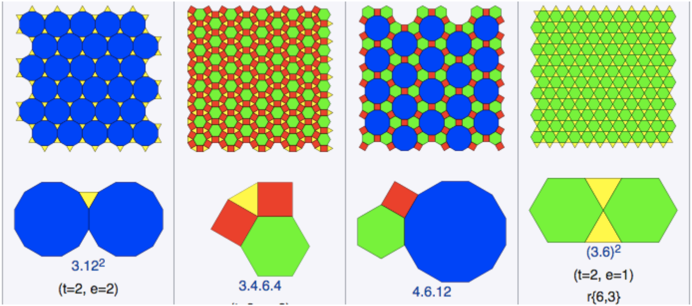

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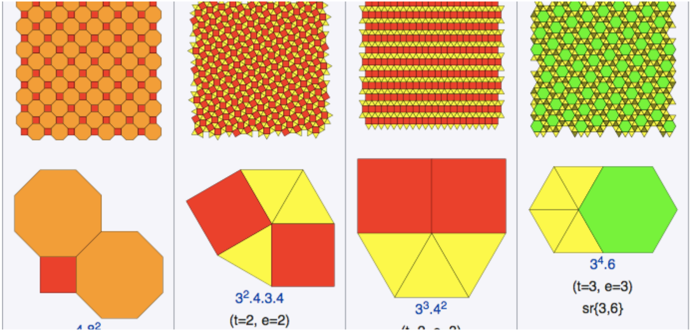

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- For $p \in [0,1]$, Bernoulli(p) percolation is automorphism-invariant.
- For $p \in (0,1)$, Bernoulli(p) percolation is insertion-tolerant and deletion tolerant.

Non-amenable One-ended Graphs

Theorem

(Z.Li, 2020)Let $G = (V, E)$ be a non-amenable, locally finite, planar, transitive graph with one end. Consider an automorphism-invariant site percolation measure μ on G with sample space $\{0,1\}^V$. Let s_0 (resp. s_1) be the total number of infinite 0-clusters (resp. infinite 1-clusters) in ω, then

$$\mu((s_0, s_1) = (1, 1)) = 0.$$

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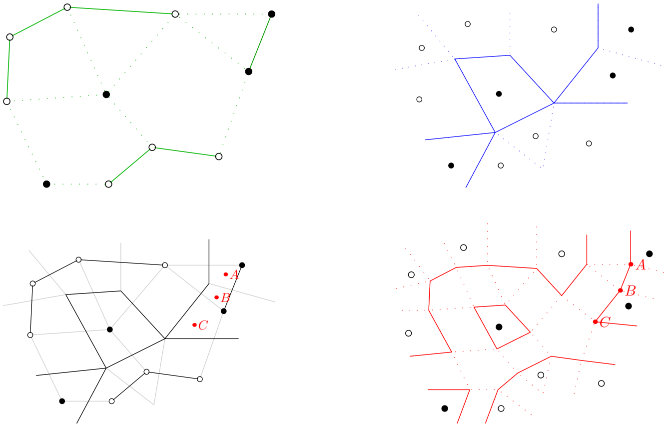
2. *If μ is deletion-tolerant, and μ-a.s. there is a unique infinite 1-cluster, then μ-a.s. there are no infinite 0-clusters.*

Sketch of Proofs



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- L forms an invariant bond percolation on $\overline{G}^+$ which has one infinite component a.s.

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- (Benjamini-Schramm, 2000)Let G be a non-amenable, quasi-transitive, unimodular graph, and let L be an invariant percolation on G which has a single component a.s. Then $p_c(L) < 1$ a.s.

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- Contradiction to the fact that L is a doubly infinite self-avoiding path implies that $\mu((s_0, s_1) = (1, 1)) = 0$.

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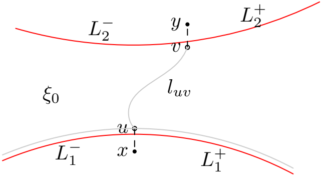
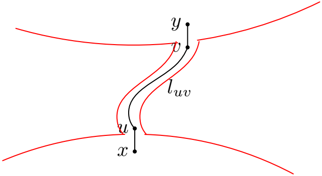
- By insertion toleration, with positive probability there exists at least two infinite 0-clusters.





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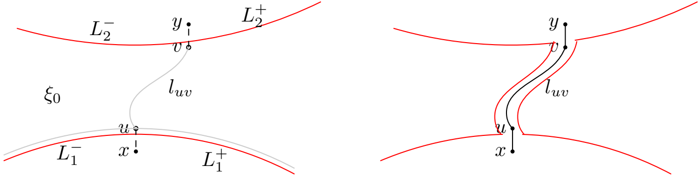

- The contradiction implies that if a.s. there is a unique infinite 0-cluster, then a.s. there are no infinite 1-clusters for insertion tolerant percolation.

Ising Model

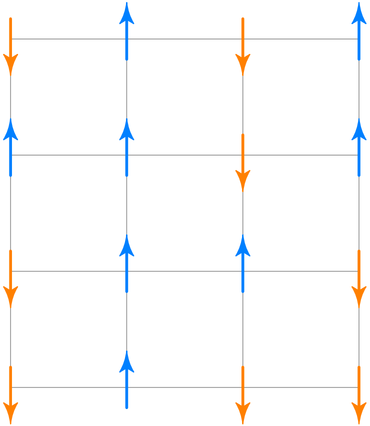

Figure: Ising Model on the Square Grid

XOR Ising Model

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- Contours: present edges of $\mathbb{L}_2$ separating different states of σ_{XOR}.

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- $\sigma_{XOR}(v) = \sigma_1(v)\sigma_2(v)$, for $v \in V(\mathbb{L}_1)$.
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XOR Ising Model

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- (B. de Tiliere and C. Boutilier) Contours of $\sigma_{XOR} \stackrel{d}{=}$ level lines of dimer height functions on the square-octagon lattice

Applications to the Ising Model

Theorem

Let G be a connected, vertex-transitive, locally finite, planar, nonamenable graph with one end and vertex degree $d \geq 3$. Assume $p_c^{site}(G) < \frac{1}{2}$ Consider the random Ising configuration $\omega \in \{\pm 1\}^V$ with coupling constant $J \geq 0$ on each edge. Let $h > 0$ satisfy $\frac{e^{-h}}{e^h + e^{-h}} = p_c^{site}(G)$. If $0 \leq J < \frac{h}{d}$, then a.s. there are infinitely many infinite "$+$"-clusters and infinitely many infinite "$-$"-clusters in ω, and infinitely many infinite contours in ϕ_ω^+, under any $\mathrm{Aut}(G)$-invariant Gibbs measure.

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- Note that when J sufficiently small, the random-cluster representation of the Ising model a.s. has no infinite clusters.

Thank you!