On boundary correlations in the Ashkin–Teller model

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Motivation

1. Understand better the connection between:
   - the classical *Pfaffian formulas* for boundary spin correlations in planar Ising models (Groeneveld & Boel & Kasteleyn, '78)
   - more recent *total positivity* inequalities for such correlations (L. '16, Galashin & Pylyavskyy '18)
2. Inspired by the idea of Aizenman & Duminil-Copin & Tassion & Warzel '18 who used *random currents* to establish Pfaffian identities for Ising correlations
3. Generalize random currents beyond the Ising model (in our case to the *Ashkin–Teller model*): switching lemmas, correlation inequalities

Outcome

1. A simple framework from which both Pfaffian identities (also for Kadanoff–Ceva order-disorder correlators) and total positivity arise: uses random currents with no reference to other models (dimers, alternating flows)

2. Linear relations for boundary spin and bulk order-disorder correlations of planar Ashkin–Teller model. In general, no Pfaffian relations since there is no factorization of correlations over the two spin configurations

3. Inequalities for correlations of Ashkin–Teller models, some light shed on dimer representations of Ising models, new proof of the classical switching lemma

Outline

1. Definition of the Ashkin–Teller model and review of previous results for the Ising model
2. Definition of random currents and two switching lemmas (with proofs): one classical and one new
3. Consequences of the second switching lemma for planar graphs: linear identities for correlations
4. Related identities for Pfaffians and determinants

The Ashkin–Teller model

Let $G = (V, E)$ be a finite (not necessarily planar) graph.

The *Ashkin-Teller model* ('43) is a probability measure on pairs of spin configurations $(\sigma, \tilde{\sigma}) \in \{-1,1\}^V \times \{-1,1\}^V$ given by

$$\mathbf{P}(\sigma, \tilde{\sigma}) = \frac{1}{\mathcal{Z}} \prod_{uv \in E} \exp\big(J_{uv}(\sigma_u\sigma_v + \tilde{\sigma}_u\tilde{\sigma}_v) + U_{uv}(\sigma_u\sigma_v\tilde{\sigma}_u\tilde{\sigma}_v + 1)\big).$$

Note that the case $U = 0$ is equivalent to two *independent Ising models*.

For $A, B \subseteq V$, let $\sigma_A = \prod_{v \in A} \sigma_v$, $\tilde{\sigma}_B = \prod_{v \in B} \tilde{\sigma}_v$, and let the *spin correlation function* is

$$\langle \sigma_A \tilde{\sigma}_B \rangle = \sum_{\sigma, \tilde{\sigma} \in \{-1,1\}^V} \sigma_A \tilde{\sigma}_B \mathbf{P}(\sigma, \tilde{\sigma}).$$

Pfaffians in the Ising model

Theorem (Groeneveld, Boel and Kasteleyn, ’79)

Let G be a planar graph and let $v_1, \ldots, v_{2k}$ be vertices placed counterclockwise on its outer boundary. Then for $U = 0$, we have

$$\langle \sigma_1 \cdots \sigma_{2k} \rangle = \mathrm{Pf}\Big(\big[\mathbf{1}\{i = j\} (-1)^{\mathbf{1}\{i<j\}} \langle \sigma_{v_i} \sigma_{v_j} \rangle \big]_{1 \leq i,j \leq 2n} \Big).$$

- manifestation of the fermionic nature of the Ising model (Majorana fermions)
- also true for Kadanoff–Ceva order–disorder insertions in the bulk
- can be proved using the exact solution of Ising in terms of dimers (e.g. on the Fisher graph) or Kac–Ward matries (L. ’14, Chelkak, Cimasoni & Kassel ’15), or through random currents and the switching lemma (Aizenman, Duminil-Copin, Tassion & Warzel ’18)

Total positivity in the Ising model

Theorem (L. '16)

Let G be a planar graph and let $s_1, s_2, \ldots, s_n, t_n, t_{n-1}, \ldots t_1$ be vertices place counterclockwise on its outer boundary. Then for $U = 0$, the matrix

$$M_{i,j} = \langle \sigma_{s_i} \sigma_{t_j} \rangle, \qquad 1 \leq i, j \leq n$$

is totally nonnegative. In particular $\det(M) \geq 0$.

- determinant of M has an interpretation as a probability of a certain topological event in the (double) random current model
- can be proved using alternating flows (L. '17) or exact bosonization of Dubédat (Galashin & Pylyavskyy '18)
- Galashin & Pylyavskyy established a bijection with the orthogonal positive Grassmannian

Currents

For a set of edges η, let $\delta(\eta)$ be the set of vertices of odd degree in the graph (η, V).

We say that a pair $\mathbf{n} = (\omega, \eta)$, where $\omega, \eta \subseteq E$, is a *current* with *sources* A if

- $\eta \subseteq \omega$
- $\delta(\eta) = A$

One can think of $\omega \setminus \eta$ as the edges with *nonzero even* values of the current, and η as the *odd valued* edges. We say that ω is the set of *open* edges of $\mathbf{n}$.

We write Ω_A for the set of all currents with sources A.

Currents

We define the *Ashkin–Teller weight* of a current by

$$w(\mathbf{n}) = 2^{k(\omega)} \prod_{e \in \eta} x_e \prod_{e \in \omega \setminus \eta} y_e,$$

where $k(\mathbf{n})$ is the number of connected components of the graph (ω, V) including isolated vertices, and where

$$x_e = e^{2U_e} \sinh(2J_e) \qquad \text{and} \qquad y_e = e^{2U_e} \cosh(2J_e) - 1.$$

Note that if $J_e \geq 0$, and

$$\cosh(2J_e) \geq e^{-2U_e},$$

then the weights are nonnegative.

We write $Z_\emptyset = \sum_{\mathbf{n} \in \Omega_\emptyset} w(\mathbf{n})$ for the partition function of sourceless currents

Switching lemma I

Switching lemma for σ and $\tilde{\sigma}$ (L. ’20)

Let $A, B \subseteq V$. Then

$$\langle \sigma_A \tilde{\sigma}_B \rangle = \frac{1}{Z_\emptyset} \sum_{\mathbf{n} \in \Omega_{A \triangle B}} w(\mathbf{n}) \mathbf{1}\{\mathbf{n} \in \mathfrak{F}_B\},$$

where $(\omega, \eta) \in \mathfrak{F}_B$ if and only if each connected component (ω, V) contains an *even* number of vertices from B.

- If $\mathbf{n} \in \Omega_A$, then $\mathbf{n} \in \mathfrak{F}_A$.
- a generalization of the switching lemma of Griffiths, Hurst, and Sherman ’70 ($U = 0$) to the Ashkin–Teller model
- a related idea appeared also in the work of Chayes and Shtengel ’00
- can be thought of as a mixed FK representation for the spin $\tau = \sigma\sigma'$ and high-temperature expansion for σ

Proof.

Consider the spin $\tau_v = \sigma_v \tilde{\sigma}_v$. We have

$$\exp\big(J_{uv}(\sigma_u\sigma_v + \tilde{\sigma}_u\tilde{\sigma}_v) + U_{uv}(\sigma_u\sigma_v\tilde{\sigma}_u\tilde{\sigma}_v + 1)\big) = 1 + \delta_{\tau_u\tau_v}(x_{uv}\sigma_u\sigma_v + y_{uv}),$$

and therefore

$$\begin{aligned}
\mathcal{Z}\langle\sigma_A\tilde{\sigma}_B\rangle &= \mathcal{Z}\langle\sigma_{A\triangle B}\tau_B\rangle = \sum_{\sigma,\tau} \sigma_{A\triangle B}\tau_B \prod_{uv\in E}\big(1 + \delta_{\tau_u\tau_v}(x_{uv}\sigma_u\sigma_v + y_{uv})\big) \\
&= \sum_{\sigma,\tau} \sigma_{A\triangle B}\tau_B \sum_{\omega\subseteq E}\prod_{uv\in\omega} \delta_{\tau_u\tau_v}(x_{uv}\sigma_u\sigma_v + y_{uv}) \\
&= \sum_{\sigma}\sum_{\omega\subseteq E} \mathbf{1}\{\omega\in\mathfrak{F}_B\} 2^{k(\omega)} \sigma_{A\triangle B} \prod_{uv\in\omega}(x_{uv}\sigma_u\sigma_v + y_{uv}) \\
&= \sum_{\sigma}\sum_{\omega\subseteq E} \mathbf{1}\{\omega\in\mathfrak{F}_B\} 2^{k(\omega)} \sum_{\eta\subseteq\omega} \sigma_{A\triangle B\triangle\delta(\eta)} \prod_{e\in\eta} x_e \prod_{e\in\omega\setminus\eta} y_e \\
&= 2^{|V|} \sum_{\omega\subseteq E} \sum_{\substack{\eta\subseteq\omega \\ \delta(\eta)=A\triangle B}} \mathbf{1}\{\omega\in\mathfrak{F}_B\} 2^{k(\omega)} \prod_{e\in\eta} x_e \prod_{e\in\omega\setminus\eta} y_e \\
&= 2^{|V|} \sum_{\mathbf{n}\in\Omega_{A\triangle B}} w(\mathbf{n}) \mathbf{1}\{\omega\in\mathfrak{F}_B\}. \qquad \square
\end{aligned}$$

A change of variables

We now turn to correlation functions of the variables $\varphi, \tilde{\varphi} \in \{-1, 0, 1\}$ defined by

$$\varphi_v = \frac{\sigma_v + \tilde{\sigma}_v}{2} \qquad \text{and} \qquad \tilde{\varphi}_v = \frac{\sigma_v - \tilde{\sigma}_v}{2}.$$

We write $\varphi_A = \prod_{v \in A} \varphi_v$, and $\tilde{\varphi}_A = \prod_{v \in A} \tilde{\varphi}_v$.

Note that $\varphi^3 = \varphi$ and $\tilde{\varphi}^3 = \varphi$, and hence it is enough to look at correlations of order at most two. Also note that $\varphi\tilde{\varphi} = 0$, and we only need to consider insertions of φ and $\tilde{\varphi}$ at disjoint sets of vertices.

It turns out that correlations of φ and $\tilde{\varphi}$ have a natural representation in terms of currents as well.

Switching lemma II

Switching lemma for φ and $\tilde{\varphi}$ (L. '20)

Let $A = A_1 \cup A_2, B = B_1 \cup B_2 \subseteq V$ be such that A_1, A_2, B_1, B_2 are pairwise disjoint. Then

$$\langle \varphi_{A_1} \varphi_{A_2}^2 \tilde{\varphi}_{B_1} \tilde{\varphi}_{B_2}^2 \rangle = \frac{1}{Z_\emptyset} \sum_{\mathbf{n} \in \Omega_{A_1 \cup B_1}} w(\mathbf{n}) 2^{-k_{A\cup B}(\mathbf{n})} \mathbf{1}\{A \overset{\mathbf{n}}{\not\leftrightarrow} B\},$$

where

- $k_{A\cup B}(\mathbf{n})$ is the number of clusters of $\mathbf{n}$ intersecting $A \cup B$
- $A \overset{\mathbf{n}}{\not\leftrightarrow} B$ means that no vertex in A is connected to a vertex in B via a path of open edges of $\mathbf{n}$

These correlators

- can be forced to vanish by means of (planar) topology. *This will yield Pfaffian relations*
- are nonnegative (first Griffiths inequality). *This will yield total positivity*

Proof.

Let $\mathcal{P}(A \cup B)$ be the set of subsets of $A \cup B$ of even cardinality. Using that $\varphi_v^2 = \sigma_v \varphi_v$ and $\varphi_v^2 = \sigma_v \tilde{\varphi}_v$ we have

$$2^{|A\cup B|} \langle \varphi_{A_1} \varphi_{A_2}^2 \tilde{\varphi}_{B_1} \tilde{\varphi}_{B_2}^2 \rangle = \sum_{S \in \mathcal{P}(A\cup B)} (-1)^{|S\cap B|} \langle \sigma_{(A_1 \cup B_1)\triangle S} \tilde{\sigma}_S \rangle$$

By the switching lemma applied to each term on the right-hand side, it is enough to show that for all $\mathbf{n} \in \Omega_{A_1 \cup B_1}$,

$$\sum_{S \in \mathcal{P}(A\cup B)} (-1)^{|S\cap B|} \mathbf{1}\{\mathbf{n} \in \mathfrak{F}_S\} = 2^{|A\cup B| - k_{A\cup B}(\mathbf{n})} \mathbf{1}\{A \overset{\mathbf{n}}{\nleftrightarrow} B\}.$$

Indeed,

- if $A \overset{\mathbf{n}}{\nleftrightarrow} B$, then the sign in the sum is constant and equal to one. This accounts for the factor $2^{|A\cup B| - k_{A\cup B}(\mathbf{n})}$ which is the number of sets $S \in \mathcal{P}(A \cup B)$ such that $\mathbf{n} \in \mathfrak{F}_S$.
- if $A \overset{\mathbf{n}}{\leftrightarrow} B$, take $u \in A$ and $v \in B$ in the same connected component of ω. Then $S \mapsto S \triangle \{u, v\}$ is a *sign-reversing involution* on sets satisfying $S \in \mathcal{P}(A \cup B)$ and $\mathbf{n} \in \mathfrak{F}_S$.

□

Planarity

We consider a finite connected *planar* graph $G = (V, E)$ embedded in the plane.

We fix a set $A_1 \cup A_2 \cup B_1 \cup B_2$ of vertices on the *outer face* of G. We will study correlation functions of the form

$$\langle \varphi_{A_1} \varphi_{A_2}^2 \tilde{\varphi}_{B_1} \tilde{\varphi}_{B_2}^2 \rangle.$$

The vertices in $A_2 \cup B_2$ will be referred to as *doubled*.

Planarity

We define $\mathcal{N}$ to be the set of vertices with multiplicities, i.e. where each doubled vertex is included as two copies $v^<$ and $v^>$. We will refer to the elements of $\mathcal{N}$ as *nodes*.

We assume that $|\mathcal{N}|$ is even (otherwise the correlation function vanishes), and fix a counterclockwise order

$$v_1, v_2, \ldots, v_{2n}$$

on $\mathcal{N}$ which agrees with the placement of the nodes on the boundary, and where for each doubled node v, the copy $v^<$ comes before $v^>$.

We split the nodes $\mathcal{N}$ into *even* $\mathcal{N}_e$ and *odd* $\mathcal{N}_o$ according to the index in this order.

Planarity

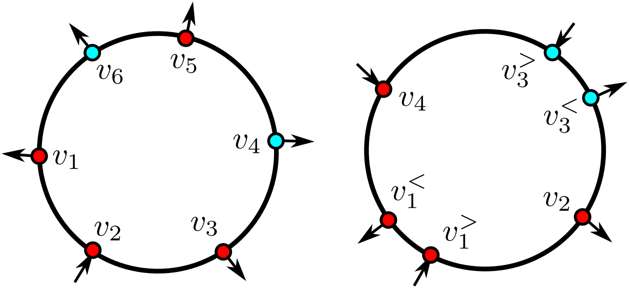
We will think of each node in $\mathcal{N}$ that comes from A (resp. B) as *red* (resp. *blue*). We write $\mathcal{R}$ and $\mathcal{B}$ for the set of red and blue nodes. We will say that the choice of $\mathcal{B}$ (and hence automatically $\mathcal{R} = \mathcal{N} \setminus \mathcal{B}$) is a *coloring* of the nodes.

Finally we partition $\mathcal{N}$ into *sources* $\mathcal{S}_+$ and *sinks* $\mathcal{S}_-$ by the rule

$$\mathcal{S}_+ = (\mathcal{N}_o \cap \mathcal{B}) \cup (\mathcal{N}_e \cap \mathcal{R}) \quad \text{and} \quad \mathcal{S}_- = (\mathcal{N}_e \cap \mathcal{B}) \cup (\mathcal{N}_o \cap \mathcal{R}).$$

In other words, as one goes along the boundary, the nodes alternate between sources and sinks as long as their color does not change, and they keep the same orientation (sink or source) whenever the color changes.

Planarity



Left: an unbalanced coloring corresponding to $\langle \varphi_{v_1}\varphi_{v_2}\varphi_{v_3}\tilde{\varphi}_{v_4}\varphi_{v_5}\tilde{\varphi}_{v_6}\rangle$. The arrows pointing inside (resp. outside) represent sources (resp. sinks).

Right: a balanced coloring corresponding to $\langle \varphi_{v_1}^2\varphi_{v_2}\tilde{\varphi}_{v_3}^2\varphi_{v_4}\rangle$. Here $W = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{N} = \{v_1^<, v_1^>, v_2, v_3^<, v_3^>, v_4\}$.

Planarity

We will also consider *planar (noncrossing)* partitions π of $\mathcal{N}$.

A partition is called *even* if each of its components contains an even number of nodes (possibly zero).

We say that that π is *compatible with* a coloring $\mathcal{B}$ if all nodes in every component of π are of the same color.

A coloring $\mathcal{B}$ is *realizable* if there exists an even planar partition that is compatible with $\mathcal{B}$.

A coloring $\mathcal{B}$ is *balanced* if the numbers of resulting sources and sinks are equal.

Lemma

A coloring $\mathcal{B}$ is realizable if and only if it is balanced.

Corollary: Unbalanced colorings

Let $\mathcal{B}$ be an unbalanced coloring of $\mathcal{N}$. Then,

$$\langle \varphi_{A_1} \varphi_{A_2}^2 \tilde{\varphi}_{B_1} \tilde{\varphi}_{B_2}^2 \rangle = \frac{1}{2^{|\mathcal{N}|}} \sum_{S \in \mathcal{P}(\mathcal{N})} (-1)^{|S \cap \mathcal{B}|} \langle \sigma_{\mathcal{N} \setminus S} \tilde{\sigma}_S \rangle = 0.$$

Here, when evaluating the correlation function we naturally project the nodes from $\mathcal{N}$ onto the vertices in W, i.e., $\sigma_{v^<} = \sigma_{v^>} = \sigma_v$ for every doubled vertex v.

Noninteracting case

If $U = 0$, then σ and $\tilde{\sigma}$ are independent and identically distributed, and in particular

$$\langle \sigma_{\mathcal{N} \setminus S} \tilde{\sigma}_S \rangle = \langle \sigma_{\mathcal{N} \setminus S} \rangle \langle \sigma_S \rangle.$$

Moreover, if $|\mathcal{B}|$ is even, then we have

$$2 \langle \sigma_{\mathcal{N}} \rangle = \sum_{S \in \mathcal{P}(\mathcal{N}) \setminus \{\emptyset, \mathcal{N}\}} (-1)^{|S \cap \mathcal{B}|+1} \langle \sigma_S \rangle \langle \sigma_{\mathcal{N} \setminus S} \rangle.$$

This identity is also satisfied by Pfaffians!

Corollary: A balanced parallel coloring

Let $\mathcal{S}_+ = \{s_1, s_2, \ldots, s_n\}$ and $\mathcal{S}_- = \{t_1, t_2, \ldots t_n\}$ be a partition of $\mathcal{N}$ such that

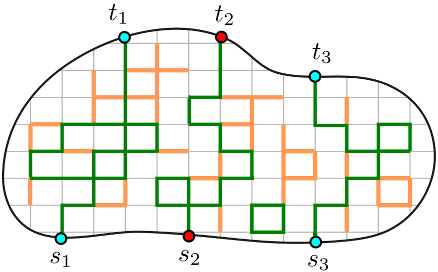
$$s_1, s_2, \ldots, s_n, t_n, t_{n-1}, \ldots t_1$$

is a counterclockwise order on $\mathcal{N}$ around the outer face.

We define

$$\mathfrak{P}_{\mathcal{S}_+,\mathcal{S}_-} = \{\mathbf{n} \in \Omega_{\mathcal{N}} : s_i \overset{\mathbf{n}}{\longleftrightarrow} t_i \text{ for all } i,\ s_i \overset{\mathbf{n}}{\not\longleftrightarrow} t_j \text{ for all } i \neq j\},$$

where $s_i \overset{\mathbf{n}}{\longleftrightarrow} t_j$ (resp. $s_i \overset{\mathbf{n}}{\not\longleftrightarrow} t_j$) means that s_i and t_i are (resp. not) connected by a path of open edges of $\mathbf{n}$.



Corollary: A balanced parallel coloring

Then, for topological reasons we have that $\mathfrak{P}_{\mathcal{S}_+,\mathcal{S}_-} = \{\mathbf{n} \in \Omega_{\mathcal{N}} : \mathcal{R} \overset{\mathbf{n}}{\not\leftrightarrow} \mathcal{B}\}$, and therefore

$$\langle \varphi_{A_1} \varphi_{A_2}^2 \tilde{\varphi}_{B_1} \tilde{\varphi}_{B_2}^2 \rangle = \frac{1}{2^{2n}} \sum_{S \in \mathcal{P}(\mathcal{N})} (-1)^{|S \cap \mathcal{B}|} \langle \tilde{\sigma}_S \sigma_{\mathcal{N} \setminus S} \rangle = \frac{1}{Z_{\emptyset} 2^n} \sum_{\mathbf{n} \in \mathfrak{P}_{\mathcal{S}_+,\mathcal{S}_-}} w(\mathbf{n}).$$

Noninteracting case

For $U = 0$, this will yield total positivity of certain boundary correlation matrices.

Pfaffians

We will consider matrices indexed by the nodes $\mathcal{N}$ with the fixed order $v_1 < \ldots < v_{2n}$.

Let K be the $\mathcal{N} \times \mathcal{N}$ square antisymmetric matrix given by

$$K_{u,v} = \mathbf{1}\{u \neq v\}(-1)^{\mathbf{1}\{v>u\}} \langle \sigma_u \sigma_v \rangle, \qquad u, v \in \mathcal{N}.$$

Again, when evaluating the correlations we project the nodes $\mathcal{N}$ onto the vertices W. In particular, if $v \in W$ is doubled, then $\sigma_v = \sigma_{v^<} = \sigma_{v^>}$.

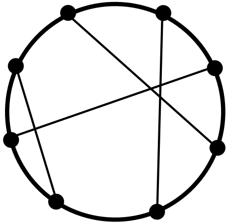
For every $S \subseteq \mathcal{N}$, we define K^S to be the restriction of K to the rows and columns indexed by S.

Pfaffians

Recall that

$$\mathrm{Pf}(K^S) = \sum_{\pi \in \Pi(S)} (-1)^{\mathrm{xg}(\pi)} \prod_{uv \in \pi} \langle \sigma_u \sigma_v \rangle,$$

where $\Pi(S)$ is the set of all pairings of S, and $\mathrm{xg}(\pi)$ is the number of crossings in π.



Here, $\mathrm{xg}(\pi) = 4$.

Determinants

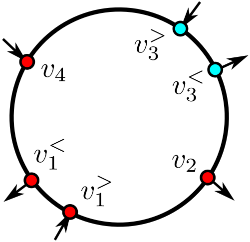
Let $\mathcal{S}_+ \cup \mathcal{S}_-$ be a partition of $\mathcal{N}$ into sources and sinks with $|\mathcal{S}_+| = |\mathcal{S}_-|$.

In analogy with Postnikov's boundary measurement matrices, we define the $\mathcal{S}_+ \times \mathcal{S}_-$ square matrix $K^{\mathcal{S}_+,\mathcal{S}_-}$ by

$$K^{\mathcal{S}_+,\mathcal{S}_-}_{u,v} = (-1)^{s(u,v)} \langle \sigma_u \sigma_v \rangle \qquad u \in \mathcal{S}_+, v \in \mathcal{S}_-, \tag{1}$$

where $s(u,v)$ is the number of sources strictly between (the smaller and the larger vertex) u and v in the fixed order on $\mathcal{N}$.

Determinants



For example, consider the correlator $\langle \varphi_{v_1}^2 \varphi_{v_2} \tilde{\varphi}_{v_3}^2 \varphi_{v_4} \rangle$ as in the figure.

Then $\mathcal{S}_+ = \{v_1^>, v_3^>, v_4\}$, $\mathcal{S}_- = \{v_1^<, v_2, v_3^<\}$ and

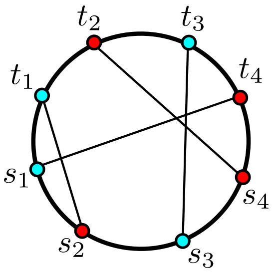
$$K^{\mathcal{S}_+,\mathcal{S}_-} = \begin{pmatrix} 1 & \langle \sigma_{v_1}\sigma_{v_2}\rangle & \langle \sigma_{v_1}\sigma_{v_3}\rangle \\ -\langle \sigma_{v_3}\sigma_{v_1}\rangle & \langle \sigma_{v_3}\sigma_{v_2}\rangle & 1 \\ \langle \sigma_{v_4}\sigma_{v_1}\rangle & -\langle \sigma_{v_4}\sigma_{v_2}\rangle & -\langle \sigma_{v_4}\sigma_{v_3}\rangle \end{pmatrix}.$$

Determinants

Let $\Pi(\mathcal{S}_+, \mathcal{S}_-) \subseteq \Pi(\mathcal{N})$ be the set of pairings in which each pair contains one source and one sink. $\Pi(\mathcal{S}_+, \mathcal{S}_-)$ can be identified with the set of bijections from $\mathcal{S}_+$ to $\mathcal{S}_-$.

An analogous formula as for Pfaffians was given for determinants by Postnikov:

$$\det(K^{\mathcal{S}_+, \mathcal{S}_-}) = \sum_{\pi \in \Pi(\mathcal{S}_+, \mathcal{S}_-)} (-1)^{\mathrm{xg}(\pi)} \prod_{uv \in \pi} \langle \sigma_u \sigma_v \rangle.$$



An identity for Pfaffians and determinants

Proposition (L. '20)

Let $\mathcal{B}$ be a coloring of $\mathcal{N}$. Then

$$\sum_{S\in\mathcal{P}(\mathcal{N})} (-1)^{|S\cap\mathcal{B}|}\,\mathrm{Pf}(K^S)\,\mathrm{Pf}(K^{\mathcal{N}\setminus S}) = \begin{cases} 2^{|\mathcal{N}|/2}\det(K^{\mathcal{S}_+,\mathcal{S}_-}) & \text{if } |\mathcal{S}_+| = |\mathcal{S}_-|, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{S}_+$ and $\mathcal{S}_-$ are the sources and sinks associated with $\mathcal{B}$.

Pfaffian formula for correlations of σ

Corollary

In direct analogy with the identity for currents, if $\mathcal{B}$ is unbalanced and $|\mathcal{B}|$ is even, then we can write

$$2\,\mathrm{Pf}(K^{\mathcal{N}}) = \sum_{S\in\mathcal{P}(\mathcal{N})\setminus\{\emptyset,\mathcal{N}\}} (-1)^{|S\cap\mathcal{B}|+1}\,\mathrm{Pf}(K^{S})\,\mathrm{Pf}(K^{\mathcal{N}\setminus S}).$$

Corollary (Pfaffian formula for Ising boundary correlations)

Let $U=0$. Then for all $S\in\mathcal{P}(\mathcal{N})$, we have

$$\langle\sigma_S\rangle = \mathrm{Pf}(K^{S}).$$

Determinantal formula for correlations of φ and $\tilde{\varphi}$

Theorem (L. '20)

Let $U = 0$. Then

$$\langle \varphi_{A_1} \varphi^2_{A_2} \tilde{\varphi}_{B_1} \tilde{\varphi}^2_{B_2} \rangle = \begin{cases} 2^{-|\mathcal{N}|/2} \det(K^{\mathcal{S}_+,\mathcal{S}_-}) & \text{if } |\mathcal{S}_+| = |\mathcal{S}_-|, \\ 0 & \text{otherwise,} \end{cases}$$

In particular the above determinant is nonnegative.

- φ's can be thought of as Dirac fermions

Total positivity of Ising boundary correlations

Let $\mathcal{S}_+, \mathcal{S}_-$ be as in the parallel coloring example. Define the $n \times n$ matrix

$$M_{i,j} = \langle \sigma_{s_i} \sigma_{t_j} \rangle, \qquad 1 \leq i,j \leq n.$$

For $I, J \subseteq \{1, \ldots, n\}$ we denote by $M^{I,J}$ the restriction of M to rows indexed by I and columns indexed by J.

Corollary (Total positivity of Ising boundary correlations)

For $U = 0$, the matrix M as defined above is totally nonnegative. Moreover, if $I, J \subseteq \{1, \ldots, n\}$ with $|I| = |J| = k$, then $\det M^{I,J} > 0$ if and only if there exist k vertex-disjoint paths in G that connect in pairs the sources from $\{s_i\}_{i \in I}$ with the sinks in $\{t_j\}_{j \in J}$.

Thank you for your attention!