Cube flips in s-embeddings and α-immersions

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joint work with Sanjay Ramassamy and Paul Thévenin

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ANR DIMERS
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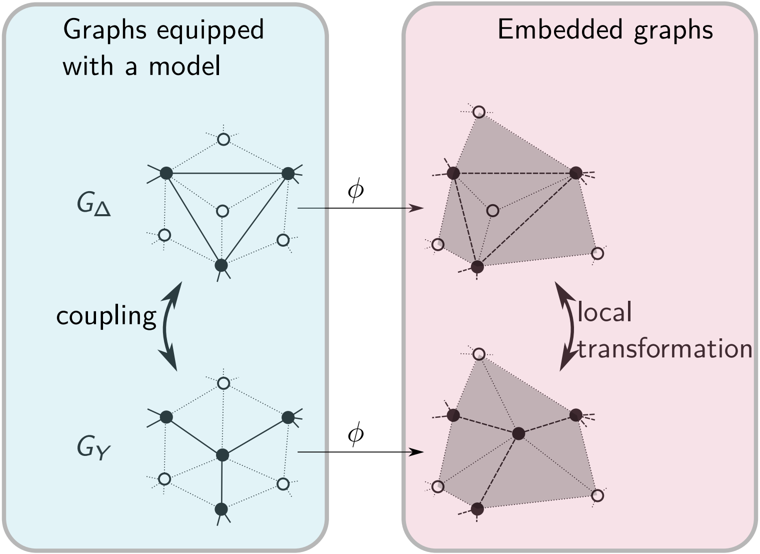
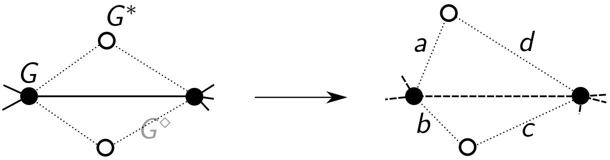
I - The cube flip property

Embeddings ϕ satisfying a *cube flip*



Example 1: isoradial embeddings

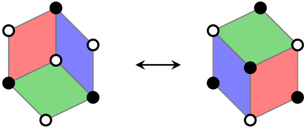
Let G be a planar graph, we denote its **dual** by G^* and its **diamond** by $G^\diamond$. In this talk we aim at embedding $G^\diamond$.



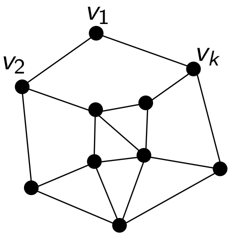
In **some** cases, $G^\diamond$ can be embedded to become a **lozenge graph**: each face of $G^\diamond$ is sent on a quadrilateral such that

$$a = b = c = d.$$

Star-triangle on G:



Example 2: Tutte embedding



Let $G = (V, E)$ be a finite planar (3-connected) graph, and $B = \{v_1, \dots, v_k\}$ be the vertices on the boundary.

For conductances $(c_e)_{e \in E}$, the corresponding laplacian is defined for $\phi : V \to \mathbb{C}$ by

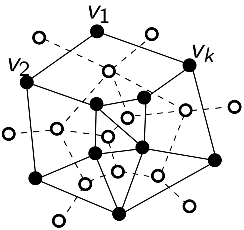
$$\Delta\phi(v) = \sum_{w \sim v} c_{vw} \left(\phi(v) - \phi(w)\right).$$

ϕ is said to be **harmonic** if $\Delta\phi(v) = 0$ for all $v \in V \setminus B$.

Theorem (Tutte, 1963)

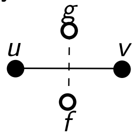
Let $z_1, \dots z_k$ be the vertices of a convex k-gon, then there exist a harmonic **embedding** ϕ such that $\phi(v_i) = z_i$

Example 2: Tutte embedding



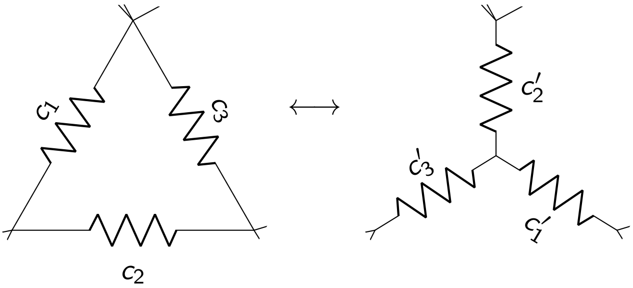
If $\phi : V \to \mathbb{C}$ is harmonic, its **dual harmonic map** $\phi^* : V^* \to \mathbb{C}$ is defined (up to the choice of the image of one point) by

$$\phi^*(g) - \phi^*(f) = i\ c_{uv}\,(\phi(v) - \phi(u))\,.$$



Then ϕ^* is harmonic on G^*,
w.r.t. the laplacian of $c^*_{fg} = \frac{1}{c_{uv}}$.
Thus we get an embedding of both G and G^*.

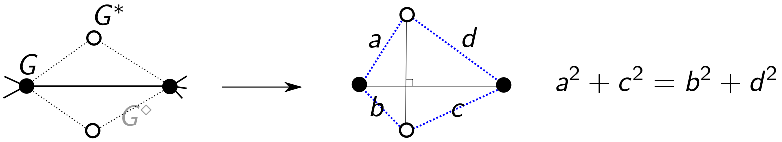
Star-triangle for conductances

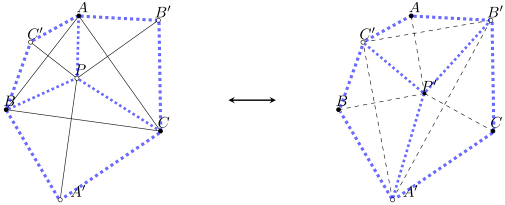


$$c_1 = \frac{c_2' c_3'}{c_1'+c_2'+c_3'} \qquad c_2 = \frac{c_1' c_3'}{c_1'+c_2'+c_3'} \qquad c_3 = \frac{c_1' c_2'}{c_1'+c_2'+c_3'}$$

...

Example 2: Tutte (orthodiagonal) embedding





Theorem *[Steiner (?); Konopelchenko-Schief '02]*

Let G_Δ, G_Y be graphs equipped with conductances related by a star-triangle move.

For any Tutte embedding of $G^\diamond_\Delta$, there exists a unique embedding of $G^\diamond_Y$ that differs only at the central point.

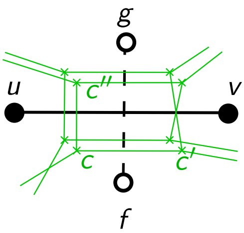
II - Cube flips for s-embeddings

S-embeddings [Chelkak '17 ...]

Now G is equipped with an Ising model with coupling constants $(J_e)_{e \in E}$. Let θ_e be s.t. $\exp(2J_e) = \frac{1+\cos\theta_e}{\sin\theta_e}$.

Propagation equation on corners:

$$X(c) = \sin\theta_e \; X(c') + \cos\theta_e \; X(c'').$$



For a complex solution X, define the s-embedding ϕ via $\phi(f) - \phi(u) = X(c)^2$.

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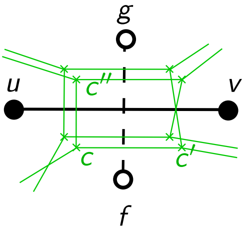
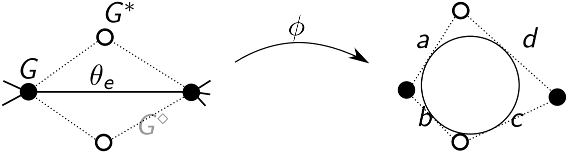


$a + c = b + d$



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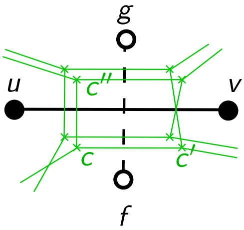

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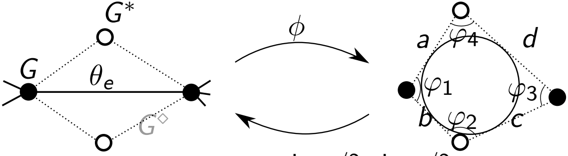
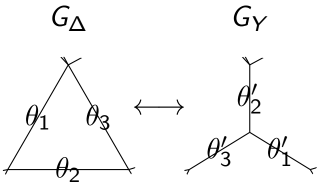


$a + c = b + d$

$$\tan^2\theta_e = \frac{\sin\varphi_1/2 \;\; \sin\varphi_3/2}{\sin\varphi_2/2 \;\; \sin\varphi_4/2}$$

Star-triangle for Ising



Theorem *[Wannier 1945, M.-Ramassamy-Thévenin 2020]*

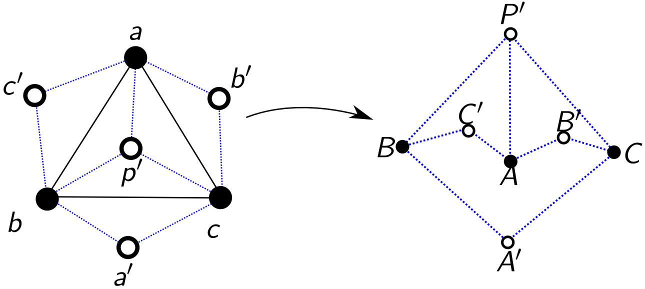
The Ising models on G_Δ and G_Y are equal in distribution if

$$\cos\theta_1' = \frac{\sin\theta_1\cos\theta_2\cos\theta_3}{\sin\theta_1+\sin\theta_2\sin\theta_3},\ \cos\theta_2' = \dots,\ \cos\theta_3' = \dots$$

$$\text{(or equivalently, } \sin\theta_1 = \frac{\cos\theta_1'\sin\theta_2'\sin\theta_3'}{\cos\theta_1'+\cos\theta_2'\cos\theta_3'},\dots\text{)}$$

The cube flip for s-embeddings

An embedding of $G^\diamond$ is said to be **proper** if the vertices around every face are ordered as in the initial graph (*map*).
Counter-example:



The cube flip for s-embeddings

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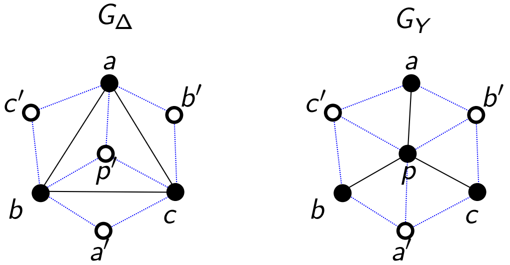
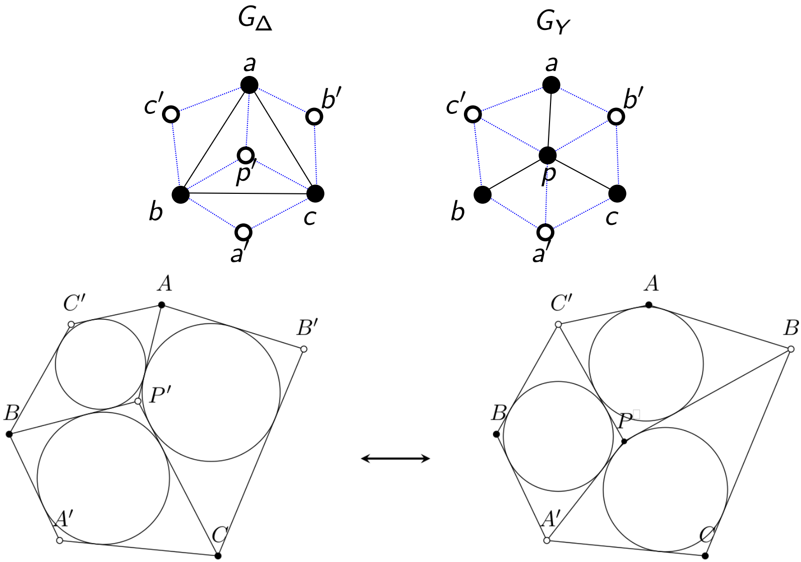
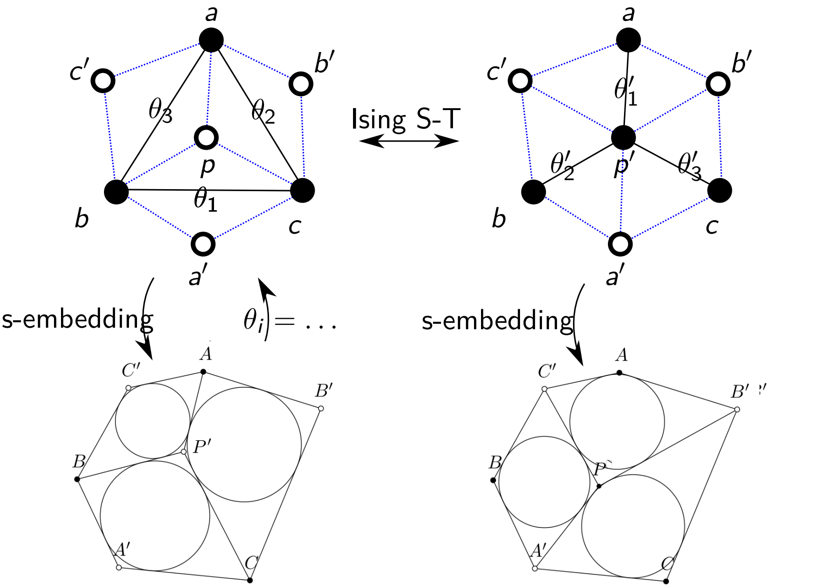


Theorem *[M.-Ramassamy-Thévenin 2020]*

For any **proper** s-embedding of $G^\diamond_\Delta$ (l.h.s), there exists a unique **proper** s-embedding of $G^\diamond_Y$ (r.h.s) that differs only at the central point.


G_Δ
G_Y
a
b
c
a′
b′
c′
p′
p
A
B
C
A′
B′
C′
P′
P

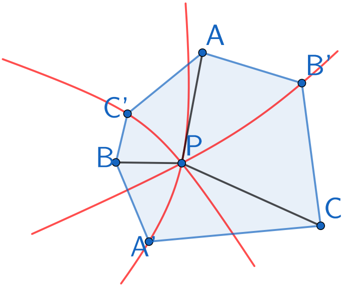
Proof - existence



Proof - uniqueness

Show that for any proper hexagon A, C', B, A', C, B',
there is at most one point P'
that gives three proper tangential quads.
If P' exists, then distances satisfy

$$P'A - P'B = C'A - C'B$$
$$P'B - P'C = A'B - A'C$$
$$P'C - P'A = B'C - B'A.$$



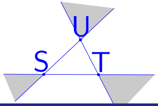
For two distinct points A, B and $\lambda \in \mathbb{R}$,
the set of points

$$\{M \mid MA - MB = \lambda\}$$

is a **branch of hyperbola**. When $\lambda > 0$, A is called the **exterior focus** and B the **interior focus**.

Proof - uniqueness

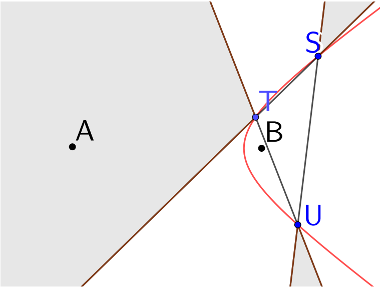
For a triangle STU, we define
the **corner chambers** as the grey open sets:



Lemma 1

If S, T, U belong to a branch of hyperbola, then the exterior focus is in an corner chamber of STU, and the interior focus is not.

(Based on convexity)



Proof - uniqueness

Lemma 2

Two distinct branches of hyperbolas that have a focus in common intersect at at most 2 points.

Proof:

$$MA - MB = \lambda,$$
$$MB - MC = \mu$$
$$\Rightarrow MA - MC = \lambda + \mu.$$

Hence we can assume that the common focus B is interior for one branch and exterior for the other.
If the branches intersect at S, T, U then the common focus is both in an corner chamber of STU and not in one. □

Consequence: there are at most 2 points P_1', P_2' for our initial problem.

Proof - uniqueness

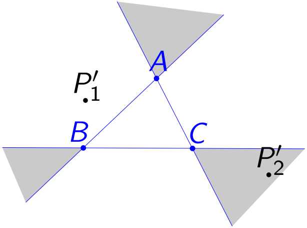
Suppose that P_1', P_2' give tangential quads:

$$P_i'A - P_i'B = C'A - C'B$$
$$P_i'B - P_i'C = A'B - A'C$$
$$P_i'C - P_i'A = B'C - B'A.$$

Then

$$P_1'A - P_2'A = P_1'B - P_2'B = P_1'C - P_2'C.$$

Hence A, B, C are on a single branch of hyperbola with foci P_1', P_2'. By Lemma 1 again, one of P_1', P_2' is in a corner chamber of ABC and the other is not.



But then the cyclic order of $\overrightarrow{P_1'A}, \overrightarrow{P_1'B}, \overrightarrow{P_1'C}$ and $\overrightarrow{P_2'A}, \overrightarrow{P_2'B}, \overrightarrow{P_2'C}$ are different.

However this order is fixed by the "proper" requirement. □

III - α-immersions

α-quads

Let $\alpha \in \mathbb{R}^*$.
A quadrilateral with successive side lengths a, b, c, d is said to be an α-quad if

$$a^\alpha + c^\alpha = b^\alpha + d^\alpha.$$

Previous cases:

- $\alpha = 1 \Leftrightarrow$ tangential;
- $\alpha = 2 \Leftrightarrow$ orthodiagonal.

This definition extends to:

- $\alpha = 0$ with $ac = bd$,
- $\alpha = -\infty$ with $\min(a, c) = \min(b, d)$,
- $\alpha = +\infty$ with $\max(a, c) = \max(b, d)$.

The space of α-quads

Proposition

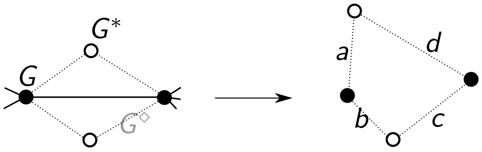
If a quadrilateral is both an α-quad and an α'-quad for $\alpha \neq \alpha'$, then it is a kite.

Proposition

A quadrilateral is an α-quad for some $\alpha \in \mathbb{R} \cup \{-\infty, +\infty\}$ *iff* its smallest and longest sides are non-adjacent.

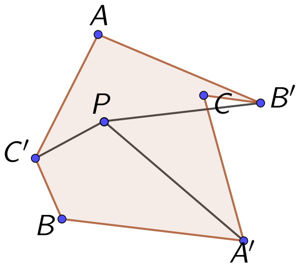
α-immersions

An α-**immersion** of $G^\diamond$ is an injective map $V \cup V^* \to \mathbb{C}$ such that faces of $G^\diamond$ are mapped on α-quads. **No embedding assumption**.

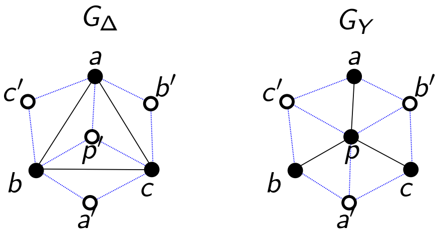


$$a^\alpha + c^\alpha = b^\alpha + d^\alpha$$

Example ($\alpha = 4$):



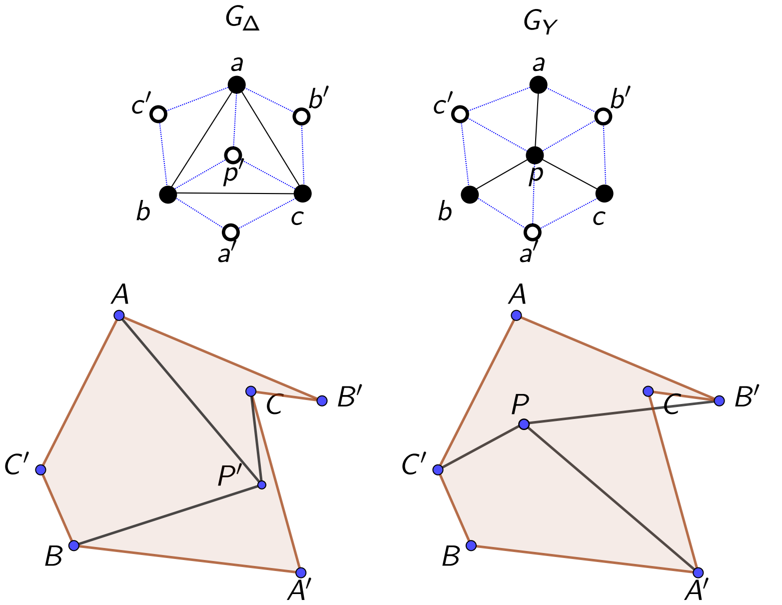
Cube flip for α-immersions, $\alpha > 1$.



Theorem *[M.-Ramassamy-Thévenin 2020]*

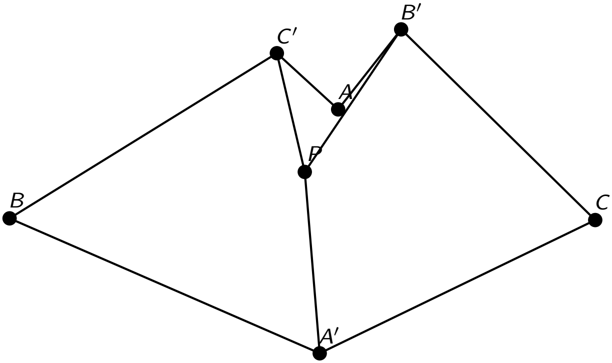
Let $\alpha > 1$.
For any α-immersion of $G^\diamond_\Delta$ (l.h.s), there exists an α-immersion of $G^\diamond_Y$ (r.h.s) that differs only at the central point.

- Expected to be **false** for $\alpha \leq 1$.
- No uniqueness.



G_Δ
G_Y
a
b
c
a′
b′
c′
p′
a
b
c
p
a′
b′
c′
A
B
C
A′
B′
C′
P′
A
B
C
P
A′
B′
C′

$\alpha = 4.3$



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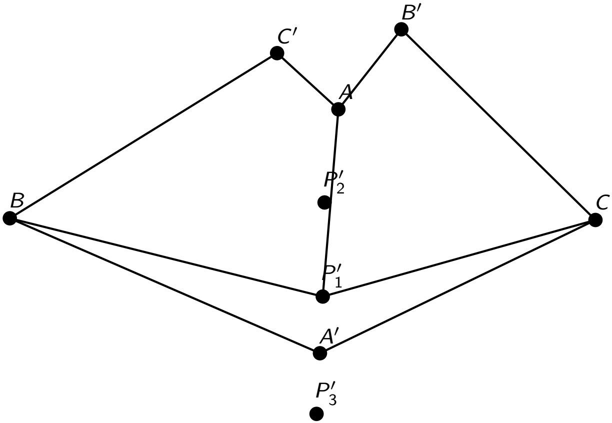

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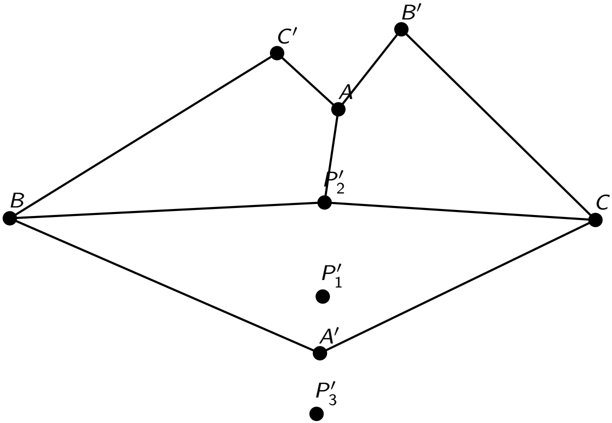

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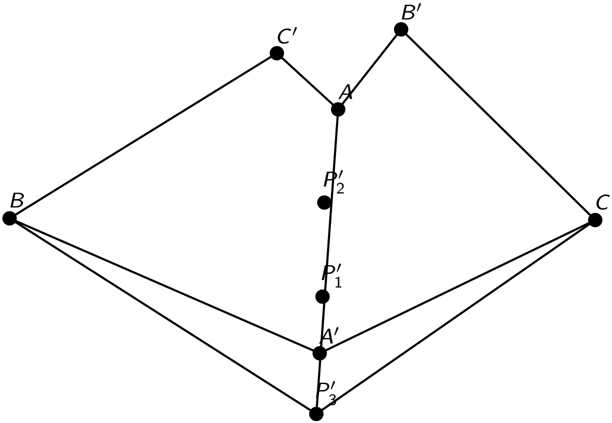
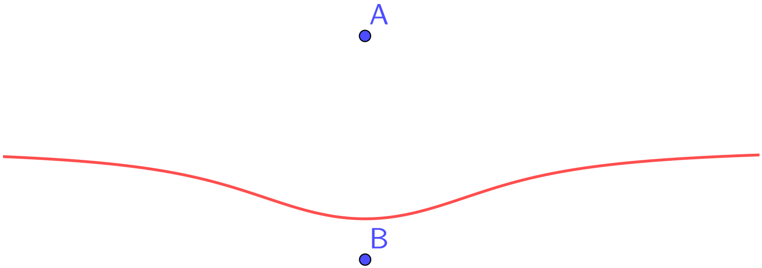


Idea of proof

For two points A, B and $\lambda \in \mathbb{R}$, the set of points

$$\{M \mid MA^\alpha - MB^\alpha = \lambda\}$$

is called an α-**curve** of foci A, B.



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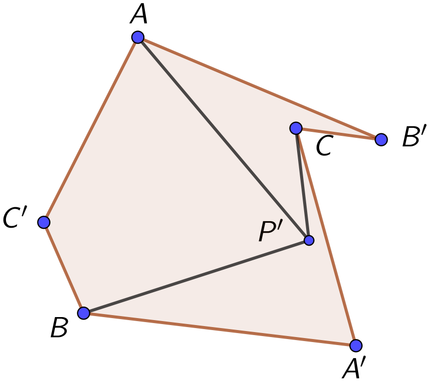
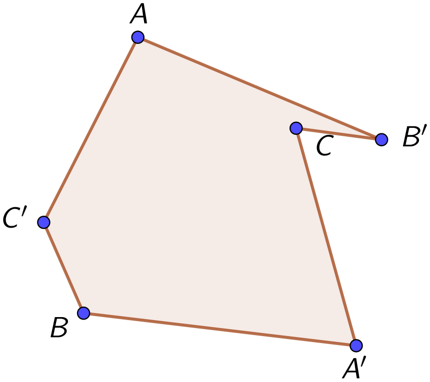


$$AC'^{\alpha} - C'B^{\alpha} + BA'^{\alpha} - A'C^{\alpha} + CB'^{\alpha} - B'A^{\alpha} = 0.$$

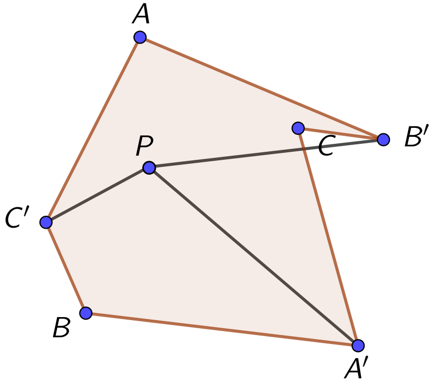
Idea of proof



$$AC'^{\alpha} - C'B^{\alpha} + BA'^{\alpha} - A'C^{\alpha} + CB'^{\alpha} - B'A^{\alpha} = 0.$$

Hence if P is at the intersection of two of the three α-curves, it is on the third.

Idea of proof



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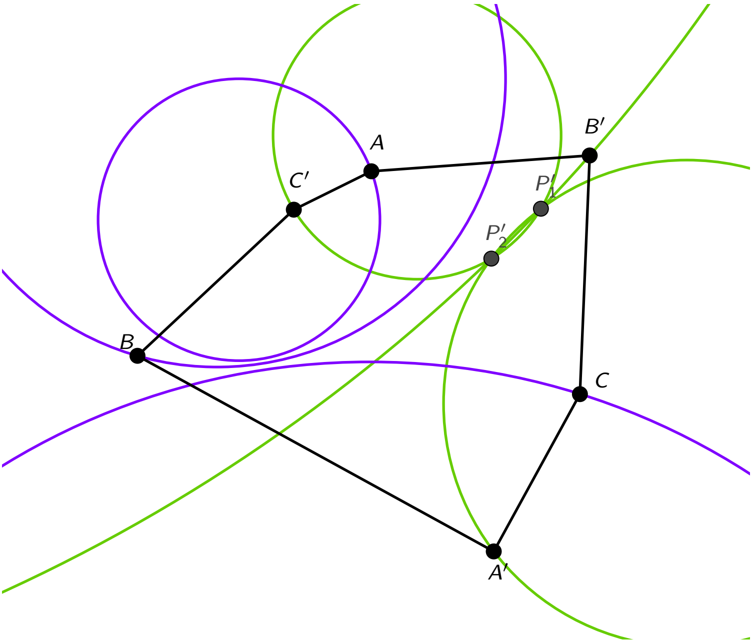
Hence if P is at the intersection of two of the three α-curves, it is on the third.
Prove that two α-curves with a common focus have to intersect (analytic proof).

$\alpha = 0$



IV - Perspectives

Other quadrilaterals?

- Conjecture: for $\alpha > 1$ there are at most 3 points of intersection.
- Do α-immersions correspond to some models / integrable systems? For some specific values of α?
- For a homogeneous symmetric function $f : \mathbb{R}^2 \to \mathbb{R}$, one can define f-quads by

$$f(a, c) = f(b, d).$$

  Are there other functions that satisfy a cube flip?

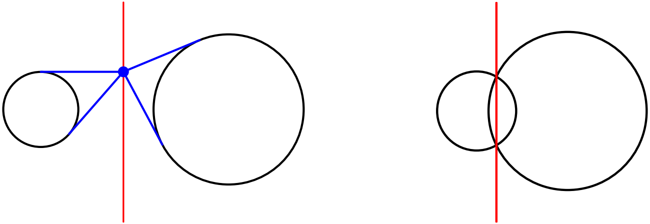
$\alpha = 0$

For $\alpha = 0$, the definition of 0-quads is $ac = bd$. In this case 0-curves

$$\{M \mid MA = \lambda MB\}$$

are **circles**.

For two circles $\mathcal{C}, \mathcal{C}'$, their **radical axis** is defined as the locus of points from which the tangents to both circles have equal lengths. When the circles intersect this is the secant line.



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Let $A_1, \ldots, A_6$, and let $\mathcal{C}_i$ be the 0-curve of foci A_{i-1}, A_{i+1} going through A_i. It seems that $\mathcal{C}_1, \mathcal{C}_3, \mathcal{C}_5$ have the same pairwise radical axis *iff* $\mathcal{C}_2, \mathcal{C}_4, \mathcal{C}_6$ do.