The Dirichlet problem for orthodiagonal maps

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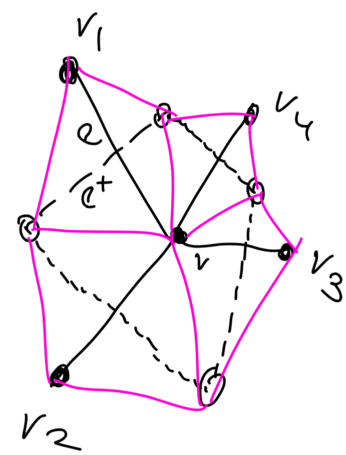
Orthodiagonal map - quadrangulation (w/ bdry) such the diagonals are orthogonal.

Comes w a nice RW on B/W

$$c(e) := \frac{\mathrm{len}(e^+)}{\mathrm{len}(e)}$$

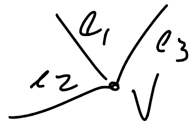


Prop RW ~~on~~ is MG

unit vec in dire $\vec{e_i}$

Proof: $v \equiv 0$

$$\sum_{i=1}^{\deg(v)} \frac{\mathrm{len}(e_i^+)}{\mathrm{len}(e_i)} \vec{e_i} \stackrel{?}{=} 0$$



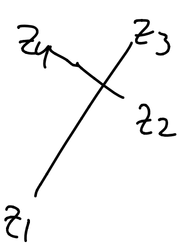
Apply rotation $\sum e_i^+ = 0$ ✓ □

<u>Result</u> "informal": On any seq of OD maps w/ mesh size $\to 0$, that approximate simply conn. domain Ω then RW $\to$ 2D BM (after time)

Earlier results by Skopenkov '13, Brent Werness same result under various regularity assum. such as bounded vertex, bounded angles

<u>Thm</u>: Seq of OD maps, ~~conv~~ approximating Ω mesh size $\to 0$, the discrete harmonic func's determined by cts boundary values $\Rightarrow$ continuous counterpart.

$\Leftarrow$ Conv. discrete hol. func's



f: disc. hol

$$\Leftrightarrow \quad \frac{f(z_1)-f(z_3)}{z_1-z_3} = \frac{f(z_2)-f(z_4)}{z_2-z_4}$$

Why OD

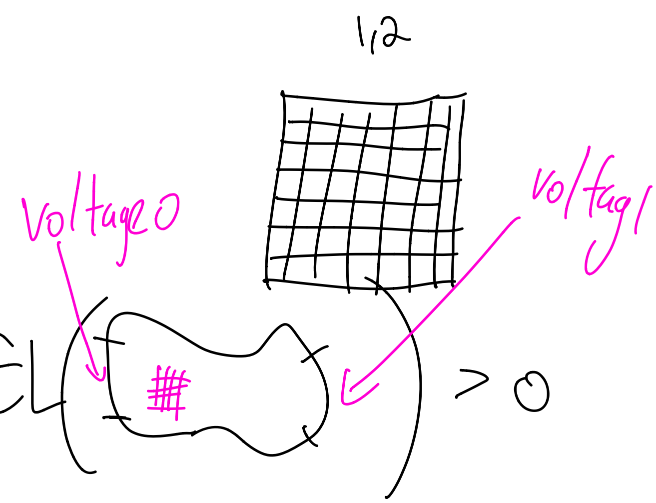
Claim: Any finite, simple, 3-conn. planar map has OD representation (map on $\mathbb{B} \cong \mathbb{M}$)

Conclusion: Random planar maps have OD rep.

Open problem: showing that mesh-size in RPMs $\to 0$.

Proof

Extremal length EL(



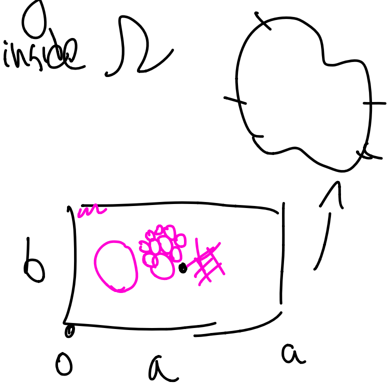
) > 0

This # is invariant under conformal maps.

$$EL = \frac{1}{\text{electric current}}$$

Thm (Ahlfors 53): $f:\Omega\to\Omega'$ oriant. pres. hom., s.t. preserves EL, then f is conformal.

of any quad inside Ω



Thm Consider a rect

OD map inside the rect

Assume: Area of ~~not~~ rect. not covered by map. is $o(1)$.

Then

$$\mathrm{EL}(\mathrm{Left}\leftrightarrow\mathrm{Right}) = \frac{b}{a} + o(1)$$

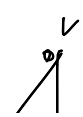
Pf:

$$\frac{1}{\mathrm{EL}} = \inf\left\{\mathcal{E}(h) : \begin{array}{l} h: B\to\mathbb{R} \\ h(\mathrm{Left})=0 \\ h(\mathrm{Right})=1 \end{array}\right\}$$

$$\mathcal{E}(h) = \sum_{(u,v)\in E} (h(v)-h(u))^2\cdot c_e$$

Take $h^\circ: B\to\mathbb{R}$ $\quad h(v) =$ horizontal coordinate

$$\mathcal{E}(h^\circ) = \sum \frac{\mathrm{len}(uv^*)}{\dots}\,\mathrm{len}(uv)^2\cos^2(\theta_{u,v})$$



$(u,v)\in E$ $\cancel{\text{len}(u,v)}$

$$= \sum_e \text{len}(e^\dagger)\,\text{len}(e)\cos^2(\theta_e)$$

white vertice

$h^\circ : W \to \mathbb{R}$ horz. coordinate

Similarly:

$$\mathcal{E}(h^\circ) = \sum_e \text{len}(e^\dagger)\,\text{len}(e)\sin^2(\theta_e)$$

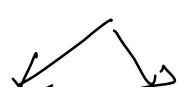
$$\Rightarrow \quad \mathcal{E}(h^\bullet) + \mathcal{E}(h^\circ) = \sum \text{len}(e)\,\text{len}(e^\dagger) = 2\,\text{Area} + o(1) = 2ab + o(1)$$

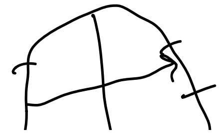
We get:

$$\boxed{\frac{1}{EL^\bullet(L\leftrightarrow R)} + \frac{1}{EL^\circ(L\leftrightarrow R)} \le \frac{2b}{a} + o(1) \qquad (*)}$$

Same thing for Top $\leftrightarrow$ Bottom

$$\frac{1}{EL^\bullet(T\leftrightarrow B)} + \frac{1}{EL^\circ(T\leftrightarrow B)} \le \frac{2a}{b} + o(1)$$





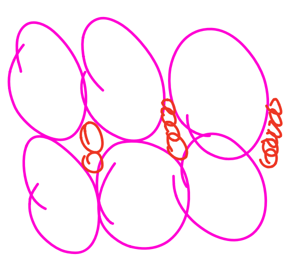
$$\Rightarrow \mathbb{E}L^{\bullet}(L \leftrightarrow R) + \mathbb{E}L^{\circ}(L \leftrightarrow R) \le \frac{2a}{b} + o(1) \qquad (**)$$

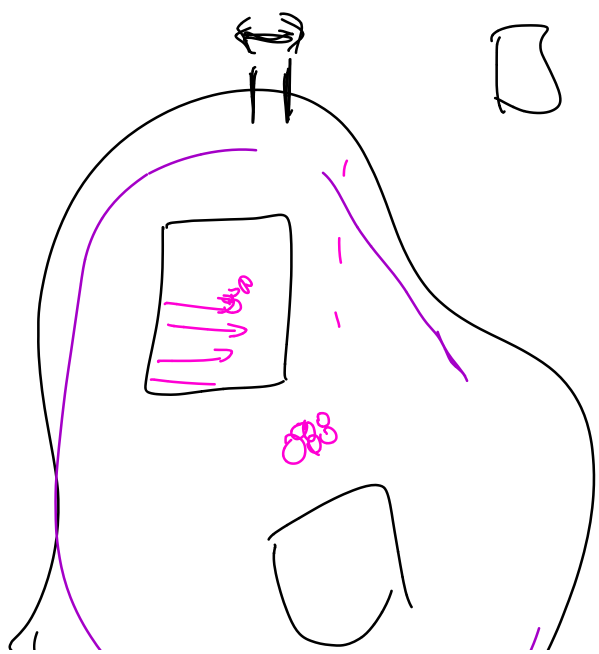
$$o(1) + \frac{a}{b} \ge \frac{\mathbb{E}L^{\bullet}(L \leftrightarrow R) + \mathbb{E}L^{\circ}(L \leftrightarrow R)}{2}$$

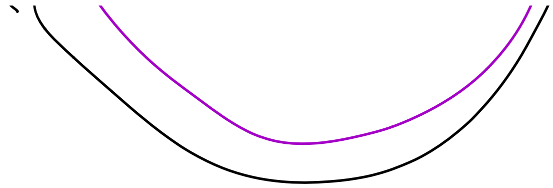
$$\ge \frac{2}{\frac{1}{\mathbb{E}L^{\bullet}(L \leftrightarrow R)} + \frac{1}{\mathbb{E}L^{\circ}(L \leftrightarrow R)}} \ge \frac{a}{b} - o(1)$$

$$\frac{x+y}{2} \ge \frac{2}{\frac{1}{x} + \frac{1}{y}}$$

$\Rightarrow$ Equality everywhere $\mathbb{E}L^{\bullet}(L \leftrightarrow R) = \frac{a}{b}$







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