Geometric shapes from discrete holomorphic functions

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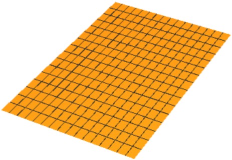
Minimal surfaces

A surface $M \subset \mathbb{R}^3$ is *minimal*, if and only of its mean curvature H is equal to zero at all points.

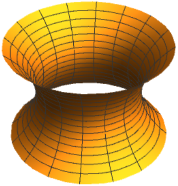
Steiner formula

Let $f : \Sigma^2 \to \mathbb{R}^3$ be a smooth surface with Gauss map $n : \Sigma^2 \to \mathbb{S}^2$. Let $f^\lambda = f + \lambda n$ denote the parallel surface at distance $\lambda \in \{-\varepsilon, \varepsilon\}$. Then

$$dA(f^\lambda) = (1 - 2H\lambda + K\lambda^2)dA(f).$$



Lagrange



Meusnier

The Weierstrass-Enneper representation

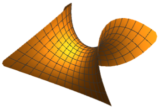
Theorem, [Weierstrass, 1866]

Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Then the surface parametrized by

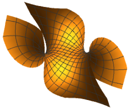
$$f(z) = \Re \int_0^z \begin{pmatrix} 1 - \varphi^2(\xi) \\ i(1 + \varphi^2(\xi)) \\ 2\varphi(\xi) \end{pmatrix} d\xi \qquad \text{(WR)}$$

is minimal.

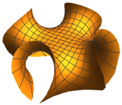
Examples:



$\varphi(z) = z$



$\varphi(z) = z^2$



$\varphi(z) = z^3$

Outline

- Discrete surfaces and isothermicity
- Gauss maps and discrete curvature
- Holomorphic functions
- Discrete Weierstrass representation

Discrete surfaces - Notation

A *discrete surface* (or *net*) is a map $f : \mathbb{Z}^2 \to V$, V a vector space (usually $\mathbb{R}^3$ today). We use the notation $f_k := f(k)$.

A *(V-valued discrete) 1-form* is a map $\alpha : \mathcal{E}(\mathbb{Z}^2) \to V$ on oriented edges of $\mathbb{Z}^2$ such that $\alpha_{kl} = -\alpha_{lk}$.

Example: A discrete surface $\psi : \mathbb{Z}^2 \to V$ induces a V-valued 1-form

$$d\psi_{kl} := \psi_l - \psi_k.$$

A 1-form is *closed* if

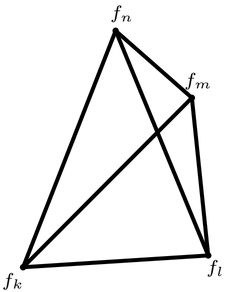
$$\alpha_{kl} + \alpha_{lm} + \alpha_{mn} + \alpha_{nk} = 0 \;\Leftrightarrow \exists \psi : \; \alpha_{kl} = d\psi_{kl}$$

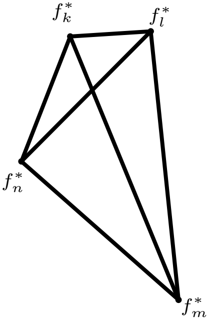
Discrete isothermic surfaces - Definition

A *discrete isothermic surface* $f : \mathbb{Z}^2 \to \mathbb{R}^3$ is a discrete surface such that for each face $(klmn) \in \mathcal{F}(\mathbb{Z}^2)$

1. the values f_k, f_l, f_m, f_n take values in a circle and
2. there exists an edge-parallel *dual surface* f^*, that is,

$$f_m - f_k \parallel f_n^* - f_l^* \text{ and } f_m^* - f_k^* \parallel f_n - f_l.$$





Discrete isothermic surfaces - Characterisation

Theorem, [Bobenko and Pinkall, 1996]

A discrete surface $f : \mathbb{Z}^2 \to \mathbb{R}^3$ is discrete isothermic if and only if there is an edge-labelling $a : \mathcal{E}(\mathbb{Z}^2) \to \mathbb{R}$ such that

$$cr(f_k, f_l, f_m, f_n) = \frac{a_{kl}}{a_{kn}} < 0.$$

Then, the dual surface is the integral of the closed 1-form df^* given by

$$df^*_{kl} = a_{kl} \frac{df_{kl}}{|df_{kl}|^2}.$$

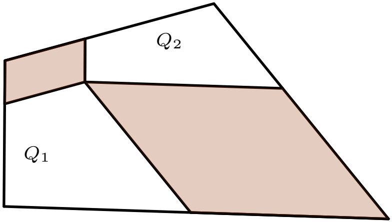
Discrete curvature - Mixed area

Let $Q = (a, b, c, d)$ be a quadrilateral in a plane. The area functional

$$A(Q) = \frac{1}{2}\det(c - a, d - b),$$

is a quadratic form on the space of edge-parallel quadrilaterals. The *mixed area* is the polar form of A. It is computed via

$$A(Q_1, Q_2) = \frac{1}{4}\left(\det(c_1 - a_1, d_2 - b_2) + \det(c_2 - a_2, d_1 - b_1)\right).$$



Discrete curvature

A circular discrete surface $f : \mathbb{Z}^2 \to \mathbb{R}^3$ allows for a *discrete Gauss map* $n : \mathbb{Z}^2 \to \mathbb{S}^2$ that is edge-parallel.

[Bobenko et al., 2010]: The discrete *mean and Gauss curvature*, H and K respectively, of a discrete surface f with respect to a discrete Gauss map n is defined as

$$H_{klmn} = -\frac{A(f,n)_{klmn}}{A(f)_{klmn}}, \quad K = \frac{A(n)_{klmn}}{A(f)_{klmn}}$$

Why is this a good definition?

Discrete Steiner formula, [Bobenko et al., 2010]

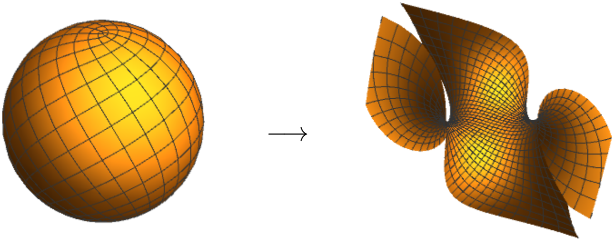
Let $f : \Sigma^2 \to \mathbb{R}^3$ be a smooth surface with Gauss map $n : \Sigma^2 \to \mathbb{S}^2$. Let $f^\lambda = f + \lambda n$ denote the parallel surface at distance $\lambda \in \{-\varepsilon, \varepsilon\}$. Then

$$A(f^\lambda)_{klmn} = (1 - 2H_{klmn}\lambda + K_{klmn}\lambda^2)A(f)_{klmn}.$$

A *discrete minimal surface* is a discrete surface $f : \mathbb{Z}^2 \to \mathbb{R}^3$ with discrete Gauss map $n : \mathbb{Z}^2 \to \mathbb{S}^2$ such that

$$
\begin{aligned}
& 0 \equiv H \\
\Leftrightarrow\ & 0 \equiv A(f, n) \\
\Leftrightarrow\ & 0 \equiv \frac{1}{4} \left(\det(f_m - f_k, n_n - n_l) + \det(n_m - n_k, f_n - f_l)\right) \\
\Leftrightarrow\ & n = f^*
\end{aligned}
$$

Recipe for discrete minimal surfaces: Produce discrete isothermic surfaces in the sphere and look for the dual surface.



Discrete holomorphic functions

Stereographic projection: $\sigma : \mathbb{S}^2 \to \mathbb{R}^2 : (x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$

A *discrete holomorphic function* is a map $\varphi : \mathbb{Z}^2 \to \mathbb{C}$ for which there exists an edge-labelling $a : \mathcal{E}(\mathbb{Z}^2) \to \mathbb{R}$ such that

$$cr(\varphi_k, \varphi_l, \varphi_m, \varphi_n) = \frac{a_{kl}}{a_{kn}} < 0.$$

Via inverse stereographic projection, a discrete holomorphic function φ is mapped to an isothermic surface in $\mathbb{S}^2$.

Discrete Weierstrass representation

Theorem [Bobenko and Pinkall, 1996]

Let $\varphi : \mathbb{Z}^2 \to \mathbb{C}$ be discrete holomorphic. Then the discrete 1-form defined by

$$df_{kl} = \Re \frac{a_{kl}}{d\varphi_{kl}} \begin{pmatrix} 1 - \varphi_k \varphi_l \\ i(1 + \varphi_k \varphi_l) \\ \varphi_k + \varphi_l \end{pmatrix},$$

is closed and descirbes a discrete minimal surface. Every discrete minimal surface arises this way.

Thank you for your attention.

References.

Bobenko, A. I. and Pinkall, U. (1996).
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