Quantum spanning forests and phase transition

Héloïse Constantin - PhD Student at UMPA

Under the supervision of Adrien Kassel,
CNRS researcher at UMPA

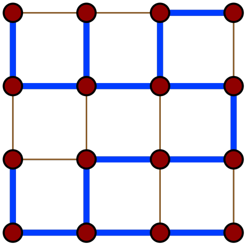
September 28, 2021

Random Spanning Tree

We call spanning tree on G a connected subgraph of G which contains all vertices of G and with no cycle.

We define a measure μ_T on spanning trees on a graph (G, E, c).

$$\mu_T(\tau) = \frac{\prod_{e\in\tau} c(e)}{Z}, Z = \sum_{\tau} \prod_{e\in\tau} c(e)$$



Random Spanning Tree

Partition function [Kirchhoff]

$$Z = \sum_{\mathcal{T}} \prod_{e \in \mathcal{T}} c(e) = \det(\Delta[r]) = \frac{1}{n} \prod_i \lambda_i$$

where λ_i are the non-zero eigenvalues of the Laplacian operator.

$$\forall f \in \mathbb{C}^V, \quad (\Delta f)(x) = \sum_{y \sim x} c(\{x, y\})(f(x) - f(y))$$

Laplacian

$$\Delta = d^*d$$

where d is the discrete derivative and d^* its adjoint.

$$d : \Omega^0(G) \to \Omega^1(G) \qquad d^* : \Omega^1(G) \to \Omega^0(G)$$

$$f \mapsto \left(df : e \mapsto f(e^+) - f(e^-)\right) \qquad \theta \mapsto \left(d^*\theta : x \mapsto \sum_{e\sim x} c(e)\theta(e)\right)$$

for the inner products

$$\langle \theta_1, \theta_2 \rangle = \sum c(e)\theta_1(e)\theta_2(e)$$

$$\langle f_1, f_2 \rangle = \sum f(x)g(x)$$

Determinantal process [Burton-Pemantle]

Edges of a random spanning tree T distributed according to μ_T form a determinantal process

$$\mathbb{P}_{\mu_T}(\{e_1, \dots, e_n\} \subset T) = \det((P(e_i, e_j))_{1 \leq i,j \leq n})$$

where

$$P = P^{\mathrm{im}(d)}_{\|\ker(d^*)}$$

Vector bundle Laplacian on a graph [Kenyon, 12]

- $G = (V, E)$ a finite graph endowed with N-dimensional Euclidean spaces $(F_x)_{x\in V}$, $(F_e)_{e\in E}$.
- ϕ a unitary connection, that is to say linear isometries $\phi_{x,e} : F_e \to F_x$ when $x \sim e$, and $\phi_{x,e} = \phi_{e,x}^{-1}$

The vector bundle Laplacian associated to the connection ϕ is a linear operator $\Delta : \Omega^0(G) \to \Omega^0(G)$

$$\Delta f(v) = \sum_{v'\sim v} c_{vv'}(f(v) - \phi_{e,v}^{-1}\phi_{e,v'} f(v')) = d^* df(v) \in F_v$$

$$d : \Omega^0(G) \to \Omega^1(G) \qquad d^* : \Omega^1(G) \to \Omega^0(G)$$

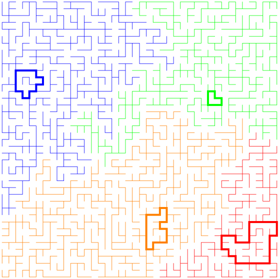
$$df(e) = \phi_{ee^+} f(e^+) - \phi_{ee^-} f(e^-) \qquad d^*\theta(v) = \sum_{v', v'v\in E} c_{v'v}\phi_{v(v'v)}\theta(v'v)$$

Cycle-rooted spanning forests

[Forman],[Kenyon, 12]

For a rank 1 bundle over a finite graph,

$$\det \Delta = \sum_{F \in CRSF} \prod_{U \text{ cycle}} \left(2 - \mathrm{hol}(U) - \frac{1}{\mathrm{hol}(U)}\right) \prod_{e \in F} c(e)$$



Measure on CRSF [Kenyon, 12]

$$\mu_\phi(F) = \frac{\prod_{e\in F} c(e) \prod_{U \text{ cycle}}(2 - \mathrm{hol}(U) - \frac{1}{\mathrm{hol}(U)})}{Z}$$

where Z is the partition function of the model

$$Z = \sum_{F\in CRSF} \prod_{e\in F} c(e) \prod_{U \text{ cycle}} (2 - \mathrm{hol}(U) - \frac{1}{\mathrm{hol}(U)}) = \det(\Delta)$$

Determinantal measure [Kenyon, 12]

Edges of a CRSF distributed according to μ_ϕ form a determinantal process with kernel

$$P = P^{\mathrm{im}(d)}_{||\ker(d^*)}$$

$$\mathbb{P}(\{e_1, ..., e_n\} \subset T) = \det((P(e_i, e_j))_{1\le i,j\le n})$$

- $G = (V, E)$ a finite graph endowed with a rank N vector bundle $(F_v)_{v \in V}, (F_e)_{e \in E}$ and a unitary connection ϕ.
- $\Delta = d^*d : \Omega^0(G) \to \Omega^0(G)$ the vector bundle Laplacian
- An orthogonal decomposition

$$\Omega^1(G) = im(d) \oplus \ker(d^*)$$

Intuition of quantum spanning forest

A random family of subspaces $(Q_e \subset F_e)_{e \in E}$ for a determinantal measure whose kernel is the orthogonal projection on

$$\mathrm{im}(d)$$

- E an inner product vector space of dimension d
- $B = (e_1, ..., e_d)$ an orthonormal basis of E
- K the matrix in this basis of a self-adjoint contraction operator k.

$$Q = \text{Vect}(e_i, i \in X)$$

where X is a finite determinantal point process associated to $S = \{1, ..., d\}$ and $K \in M_d(\mathbb{C})$.

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Determinantal linear process [A. Kassel, T. Lévy]

$$E = E_1 \oplus ... \oplus E_s$$

Concatenating uniformly sampled orthonormal bases of $E_1, ..., E_s$, and considering k a self-adjoint contraction operator gives a random subspace Q adapted to the decomposition.

Quantum spanning forest [Kassel, Lévy]

A random subspace $Q = \oplus_{e \in E} Q \cap F_e$ of

$$\Omega^1(G) = im(d) \oplus^{\perp} \ker(d^*)$$

given by the determinantal linear process associated to the orthogonal projection k on $im(d)$ and to the decomposition

$$\Omega^1(G) = \bigoplus_{e \in E_+} \Omega^1(F_e)$$

$\Omega^1(F_e) := \{w \in \Omega^1(G), w(e') = 0 \ \forall \ e' \notin \{e, -e\}\} \simeq F_e$.

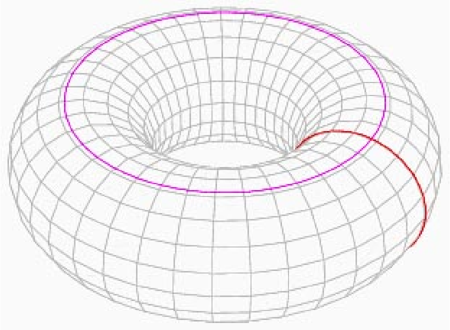
Determinantal process

$$\mathbb{P}(R \subset Q) \propto \det(k_R^R)$$

- $(G_n = \mathbb{Z}^d/(n\mathbb{Z})^d)$, $(\mathbb{C}^N)_{x\in V, e\in E}$, $(M_1, ..., M_d) \in U_N(\mathbb{C})^d$
- If $e = xy$ with $y - x = t_j = (0, ..., 0, 1, 0, ..., 0)$, define

$$\phi_{ex} = M_j, \quad \phi_{ey} = I_N$$

$$\phi_{xe} = M_j^{-1}, \quad \phi_{ye} = I_N$$



Fourier decomposition

$\Omega^0(G_n)$ and $\Omega^1(G_n)$ can be decomposed in direct sums of subspaces of periodic forms

$$\Omega^0(G_n) = \bigoplus_{(z_1,\dots,z_d)\in\mathbb{U}_n^d} E_{z_1,\dots,z_d} \qquad \Omega^1(G_n) = \bigoplus_{(z_1,\dots,z_d)\in\mathbb{U}_n^d} F_{z_1,\dots,z_d}$$

where

$$\begin{cases} E_{z_1,\dots,z_d} = \{f \in \Omega^0(G_n) \quad | \quad f(x+t_j) = z_j f(x), \quad \forall j \in [1,d]\} \\ F_{z_1,\dots,z_d} = \{\theta \in \Omega^1(G_n) \quad | \quad \theta(e+t_j) = z_j \theta(e), \quad \forall j \in [1,d]\} \end{cases}$$

Periodicity is preserved by d and d^*.

$$d(E_{z_1,\dots,z_d}) \subset F_{z_1,\dots,z_d}, \qquad d^*(F_{z_1,\dots,z_d}) \subset E_{z_1,\dots,z_d}$$

Each space $E_{z_1,\dots,z_d}$ is invariant under Δ_n and if $f \in E_{z_1,\dots,z_d}$,

$$\Delta_n f = \Delta_1(z_1, \dots, z_d) f$$

$$\Delta_1(z_1, \dots, z_d) = d^*_{|F_{z_1,\dots,z_d}} d_{|E_{z_1,\dots,z_d}}$$

There exists a determinantal measure μ_n on quantum spanning forests on G_n whose kernel is the orthogonal projection on $\mathrm{im}(d)$

$$\begin{aligned} K_n : \Omega^1(G_n) &\to \Omega^1(G_n) \\ \theta &\mapsto dGd^*\theta \end{aligned}$$

where $G = (d^* d)^{-1}_{|ker(d)^\perp}$.

Each space $F_{z_1,\dots,z_d}$ is invariant under K_n and if $\theta \in F_{z_1,\dots,z_d}$, then

$$K_n\theta = K_1(z_1, \dots, z_d)\theta$$

Writting the orthonormal basis of $\Omega^1(G_n)$ in the decomposition in Fourier spaces gives the following expression :
if $i, j \in [1, N]$, $e = [e] + (x_1, ..., x_n)$, $e' = [e'] + (y_1, ..., y_n)$

$$(K_n)^{i,j}_{e,e'} = \frac{1}{n^d} \sum_{z_1{}^n=1,...,z_d{}^n=1} \prod_{k=1}^{d} (z_k)^{y_k - x_k} (K_1(z_1, ..., z_d))^{i,j}_{[e],[e']}$$

When $n \to \infty$, for every $e, e' \in \mathbb{Z}^d$ and every $1 \leq i, j \leq N$

$$(K_n)^{i,j}_{e,e'} \to \int_{|z_1|=1,...,|z_d|=1} (K_1(z_1, ..., z_d))^{i,j}_{[e],[e']} \prod_j z_j^{y_j - x_j} \frac{dz_j}{2i\pi z_j}$$

Convergence of the sequence of measures

μ_n converges weakly to a determinantal measure on the Grassmannian of the rank N vector bundle over $\mathbb{Z}^d$ whose kernel is

$$K_{e,e'} = \int_{|z_1|=1,\dots,|z_d|=1} (K_1(z_1,\dots,z_d))_{[e],[e']} \prod_j z_j^{y_j - x_j} \frac{dz_j}{2i\pi z_j}$$

If $(z_1,\dots,z_d) \in \mathbb{T}_d$,

$$K_1(z_1,\dots,z_d) = d(z_1,\dots,z_d) G(z_1,\dots,z_d) d^*(z_1,\dots,z_d)$$

We call characteristic polynomial the polynomial

$$P(z_1,\dots,z_d) := \det(\Delta_1(z_1,\dots,z_d))$$

Regular / Singular connection

If $\forall \underline{z} \in \mathbb{T}^d, P(\underline{z}) \neq 0$, we say that the connection is regular.
Otherwise, we say that the connection is singular.

Exponential decay of correlations

If the connection is regular, we have for $i \in [1, d], \exists c_i > 0, \beta_i > 0$ such that for every $e = [e] + (x_1, ..., x_d), e' = [e'] + (y_1, ..., y_d)$,

$$|K_{e,e'}^{k,l}| \leq c_i \exp(-\beta_i |y_i - x_i|) \qquad \forall k, l \in [1, N]$$

The following properties are equivalent :

- There exists $\underline{z} = (z_1, ..., z_d) \in \mathbb{T}^d$ such that $P(\underline{z}) = 0$.
- There exists $X \in \mathbb{C}^N$ a common eigenvector to the $M_1, ..., M_d$.

Reducible connections [A. Kassel, T. Lévy]

We say that the connection is reducible if there exists sub-bundles $F^{(1)}, F^{(2)}$ of $F = (F_e)_{e \in \mathbb{Z}^d}$ and connections $h^{(1)}, h^{(2)}$ on these sub-bundles such that

$$F = F^{(1)} \oplus F^{(2)}, \qquad h = h^{(1)} \oplus h^{(2)}$$

Then the splitting $\Omega^1(\mathbb{Z}^d) = \oplus F_e$ can be refined in

$$\Omega^1(\mathbb{Z}^d) = F_e^{(1)} \oplus F_e^{(2)}.$$

If a connection given by $M_1, ..., M_d \in U_N(\mathbb{C})$ with $N > 1$ is singular, then it is reducible. The pair (F, ϕ) is isomorphic to

$$(\oplus_{1 \leq i \leq p} L^{(i)} \oplus B_{N-p} \quad , \quad \oplus_{1 \leq i \leq p} Id^{(i)} \oplus h)$$

Trace of a quantum spanning forest [A. Kassel, T. Lévy]

We call trace of a quantum spanning forest the random variable

$$n_Q = (\dim Q_e)_{e \in E}$$

If τ is a refined splitting adapted to a reducible connection h, the QSF adapted to τ is a direct sum of independant QSF adapted to $(F^{(i)}, h^{(i)}, \sigma^{(i)})$ where

$$\sigma^{(i)} : \Omega^1(\mathbb{Z}^d)^{(i)} = \oplus F_e^{(i)}$$

The law of n_Q is independant of the choice of τ.

Polynomial rate of decay of correlations

For a singular connection, $K_{e,e'}$ has a polynomial decay when $e = [e] + (x_1, ..., x_d), e' = [e'] + (y_1, ..., y_d)$, $|y_i - x_i| \to \infty$.

Diagonally-translated edges

Let $M \in U_N(\mathbb{C})$. For a singular connection $(M^{\varepsilon_1}, ..., M^{\varepsilon_d})$ with $(\varepsilon_1, ..., \varepsilon_d) \in \{\pm 1\}^d$, for all edges $e, e' = e + (\varepsilon_1 x, ..., \varepsilon_d x)$ with $x \in \mathbb{Z}$,

$$K_{e,e'} = 0_{M_N(\mathbb{C})}$$

Singularity means commuting matrices

Let $M_1, ..., M_d \in U_2(\mathbb{C})$.
The connection is singular if and only if there exists $(z_1, ..., z_d) \in \mathbb{T}^d$ such that $P(z_1, ..., z_d) = 0$ which is equivalent to the existence of a basis in which

$$M_j = \begin{pmatrix} w_j & 0 \\ 0 & \bar{z}_j \end{pmatrix} \quad \forall j \in [1, d].$$

Then the connection is singular if and only if the matrices commute.

Commutative case

If the matrices $M_1, ..., M_d \in U_2(\mathbb{C})$ commute, then the trace of the QSF Q has the same law as the sum of the traces of two independant spanning trees.

Rank 1 case with a connection with quaternionic values

[Kenyon, 2012]

For a connection in $SU_2(\mathbb{C})$, on a finite graph G_n, $\mathrm{Qdet}(\Delta_n) = \sum_{\mathsf{CRSF}} \prod_{U \in \mathsf{cycles}} (2 - \mathsf{Tr}(\mathsf{hol}(U)))$ where $\mathsf{Tr}(\mathsf{hol}(U))$ is the trace of the holonomy along the cycle.

$$\mu_n(F) = \frac{\prod_U (2 - \mathsf{Tr}(\mathsf{hol}(U)))}{\mathrm{Qdet}(\Delta_n)}$$

μ_n is a determinantal measure over CRSF of G_n.

If $A = (h_{i,j})_{N\times N}$, is a self-adjoint quaternionic matrix with

$$h_{i,j} \in \mathbb{H}, h_{i,j} = \begin{pmatrix} a_{i,j}^{1,1} & a_{i,j}^{1,2} \\ a_{i,j}^{2,1} & a_{i,j}^{2,2} \end{pmatrix} \in SU_2(\mathbb{C}),$$

$$(\mathrm{Qdet}(A))^2 = \det(a_{i,j}^{k,l})_{2N\times 2N}$$

Changing the coefficient field [A. Kassel, T. Lévy]

If $Q^{(1)}$ and $Q^{(2)}$ are independant QSF associated to k on $\mathbb{Z}^d$ with quaternionic values and C is the QSF associated to k with complex values, we have the equality in law of the traces :

$$(\dim_{\mathbb{C}} C_e)_{e\in E} = (\dim_{\mathbb{H}} Q_e^1 + \dim_{\mathbb{H}} Q_e^2)_{e\in E}$$

Work in progress

- The sequence of Q-determinantal measures μ_n associated to a periodic quaternionic connection $(h_1, \dots, h_d) \in \mathbb{U}(1, \mathbb{H})^d$ converges to a Q-determinantal measure μ on CRSF on $\mathbb{Z}^d$.
- If the quaternions $(h_1, \dots, h_d) \in \mathbb{H}^d$ do not commute, the kernel of the measure μ has an exponential rate of decay
- The measure μ gives a positive probability to configurations with cycles.

Free energy

On $G_n = \mathbb{Z}^2/(n\mathbb{Z})^2$ the reduced-determinant of the Laplacian is

$$\det{}_0(\Delta_n) = \prod_{z_1^n=1,\ldots,z_d^n=1} \det{}_0(\Delta_{G_1}(z_1, \ldots, z_d))$$

The free energy $F = \lim \frac{1}{n^d} \log(F_n)$ is

$$F = \int_{|z_1|=1,\ldots,|z_d|=1} \log(\det{}_0(\Delta_1(z_1, \ldots, z_d))) \prod \frac{dz_j}{2i\pi z_j}$$

and for almost every $z_1, \ldots, z_d$,

$$\det{}_0(\Delta_1(z_1, \ldots, z_d)) = P(z_1, \ldots, z_d) \neq 0$$

What is the non-analyticity order of the free energy ?

$$M_1 = A(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \qquad M_2 = B = \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix}$$

with $\theta \in \mathbb{R}^+, |b| = 1$.

$$AB - BA = \begin{pmatrix} 0 & 2ib\sin(\theta) \\ 2i\bar{b}\sin(\theta) & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 2ib\theta \\ 2i\bar{b}\theta & 0 \end{pmatrix}$$

Conjecture

$\frac{F_\theta - F_0}{\theta^2}$ has a finite limit when $\theta \to 0$.

Thank you for your attention!

Any question?