Ising model on random triangulations with a boundary

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Rentrée de l'ANR Dimers
28 Septembre 2021

Based on joint research with Linxiao Chen (ETHZ) and the artices [https://doi.org/10.1007/s00220-019-03672-5], [arXiv:2003.09343] and [arXiv:2003.11012]

Ising model on a graph

- Let G be a finite graph, $V(G)$ its vertex set and $E(G)$ its edge set.
- A *spin configuration* σ on G is formally defined as $\sigma = (\sigma_v)_{v \in V(G)} \in \{-1, +1\}^{V(G)}$.

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- Assign a Boltzmann measure on spin configurations by

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  where β is called the *inverse temperature*.
- Partition function $Z_G(\beta) = \sum_{\sigma} \prod_{\{v,w\} \in E(G)} e^{\beta \sigma_v \sigma_w}$

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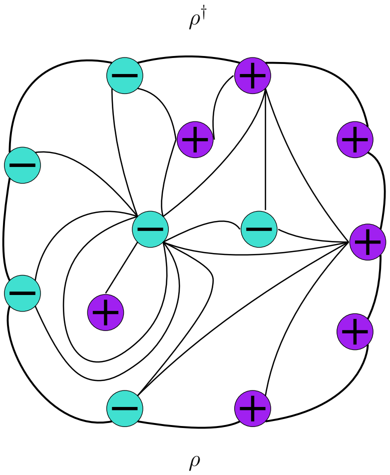
- Partition function $Z_G(\beta) = \sum_\sigma \prod_{\{v,w\} \in E(G)} e^{\beta \sigma_v \sigma_w}$
- The Boltzmann distribution can be reformulated as $\mathbb{P}_G^{\nu}(\sigma) \propto \nu^{\#\{\{v,w\} \in E(G) \,:\, \sigma_v = \sigma_w\}}$
- In partiicular, $\beta > 0 \Leftrightarrow \nu > 1$. In this regime, the model is called *ferromagnetic*, on which we concentrate in the sequel.

An example in 2d with a planar embedding



Ising model on random ("dynamical") lattices

- Dates back to the work of Kazakov (1986) and Boulatov - Kazakov (1987)
- Some physics motivations: "Liouville Quantum Gravity coupled with matter" (Polyakov 1981); quantum vs Euclidean *critical exponents* via the KPZ-relation (Knizhnik-Polyakov-Zamolodchikov 1988)

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- In the language of modern mathematics: random planar maps coupled with an (annealed) Ising model
- We want to find a critical behavior of the model which differs from the "universality class of the Brownian map", and see how it reflects to the geometry

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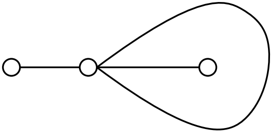

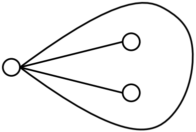

Ising-triangulations

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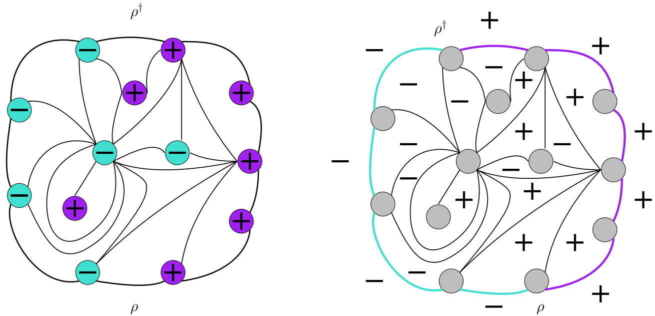
- Add to each internal face (or vertex) a spin, either $+$ or $-$.
- *Dobrushin boundary conditions*: the spins *outside* the boundary (resp. on the boundary) are fixed by a sequence of the form $+^p-^q$ counterclockwise from the root.



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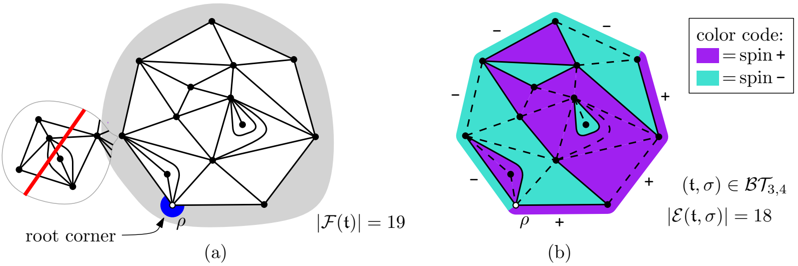
- An edge is called *monochromatic* if it separates two faces (resp. vertices) with the same spin. Let $\mathcal{E}(\mathfrak{t}, \sigma)$ be the set of monochromatic edges in $(\mathfrak{t}, \sigma)$.



Partition functions

Partition function

$$z_{p,q}(t,\nu) = \sum_{(\mathfrak{t},\sigma)\in\mathcal{BT}_{p,q}} \nu^{|\mathcal{E}(\mathfrak{t},\sigma)|} t^{\mathrm{Vol}(\mathfrak{t})},$$

where

- $\mathcal{BT}_{p,q}$ is the set of triangulations of the $(p+q)$-gon together with an Ising-configuration on either interior faces or vertices and a Dobrushin boundary condition $+^p-^q$.

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Generating function

$$Z(u,v;t,\nu) = \sum_{p,q\geq 0} z_{p,q}(t,\nu)\, u^p v^q$$

Theorem [Chen, T., 2020] (spins on faces) ([2])

For every $\nu > 1$, the GF $Z(u, v; t, \nu)$ is an algebraic function having a rational parametrization

$$t^2 = \hat{T}(S, \nu), \qquad t \cdot u = \hat{U}(H; S, \nu), \qquad t \cdot v = \hat{U}(K; S, \nu)$$
$$Z(u, v; t, \nu) = \hat{Z}(H, K; S, \nu)\,,$$

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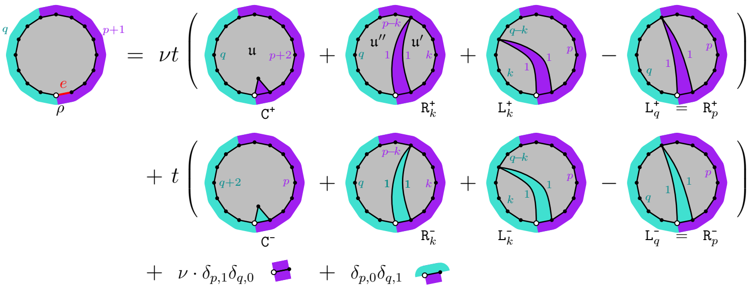
This theorem indicates that our model in concern is "exactly solvable": various observables (eg. the free energy) can be explicitly computed at least in some scaling limits with respect to the perimeter from the expression of the generating function!

Proof ingredients: peeling and functional equation for Z



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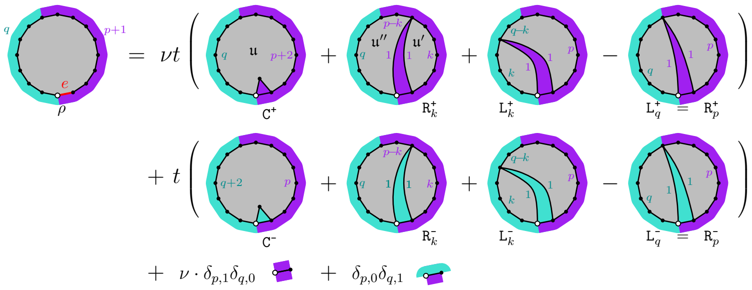

$$z_{p+1,q} = \nu t \left(z_{p+2,q} + \sum_{p_1+p_2=p} z_{p_1+1,0}\, z_{p_2+1,q} + \sum_{q_1+q_2=q} z_{1,q_1}\, z_{p+1,q_2} - z_{p+1,0}\, z_{1,q} \right)$$

$$+ t \left(z_{p,q+2} + \sum_{p_1+p_2=p} z_{p_1,1}\, z_{p_2,q+1} + \sum_{q_1+q_2=q} z_{0,q_1+1}\, z_{p,q_2+1} - z_{p,1}\, z_{0,q+1} \right)$$

$$+ \nu\, \delta_{p,1}\, \delta_{q,0} + \delta_{p,0}\, \delta_{q,1} \qquad (1)$$

Summing over p, q, we obtain a linear equation for $Z(u, v)$, and interchanging the roles of p and q gives a linear system

$$\begin{bmatrix} \Delta_u Z(u,v) \\ \Delta_v Z(v,u) \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} \nu & 1 \\ 1 & \nu \end{bmatrix} \begin{bmatrix} u + t\left(\Delta_u^2 Z(u) + (\Delta Z_0(u) + Z_1(v))\,\Delta_u Z(u) - \Delta Z_0(u) Z_1(v)\right) \\ v + t\left(\Delta_v^2 Z(v) + (\Delta Z_0(v) + Z_1(u))\,\Delta_v Z(v) - \Delta Z_0(v) Z_1(u)\right) \end{bmatrix},$$

where

$$Z_k(u) := [v^k] Z(u,v), \quad \Delta_u Z(u,v) = \frac{Z(u,v) - Z_0(v)}{u},$$

$$\Delta Z_0(u) = \frac{Z_0(u) - 1}{u}, \quad \Delta_u^2 Z(u,v) = \frac{Z(u,v) - Z_0(v) - u Z_1(v)}{u^2}$$

and so on.

By algebraic operations it turns out that Z_1 can be eliminated, and thus we obtain a rational expression

$$Z(u,v) = \frac{R_1(u,v,Z_0(u),Z_0(v))}{R_2(u,v,Z_0(u),Z_0(v))}. \tag{3}$$

where R_1, R_2 are explicit polynomials.

- Besides, we obtain a functional equation

$$\mathcal{P}(Z_0(u), u, z_1, z_3; t, \nu) = 0, \tag{4}$$

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- Luckily, we can obtain rational parametrizations for t, z_1 and z_3 by simple duality with the model in [Bernardi, Bousquet-Melou [1]]. This can also be done directly from (4) (more messy).
- Applying (computer) algebra we find an explicit RP for u and $Z_0(u)$ for any given $\nu > 1$.

Critical line

Proposition (Bernardi, Bousquet-Mélou [1])

There is a continuous decreasing function $\tau : (0, \infty) \to (0, \infty)$ *for which*

$$[t^n] z_{1,0}(t, \nu) \sim_{n \to \infty} \begin{cases} c(\nu)\tau(\nu)^{-n} n^{-5/2} & \text{if } \nu \neq \nu_c \\ c(\nu_c) t_c^{-n} n^{-7/3} & \text{if } \nu = \nu_c \end{cases}$$

where $\nu_c = 1 + 2\sqrt{7}$ *and* $t_c = \tau(\nu_c) = \frac{\sqrt{5}\sqrt{35-11\sqrt{7}}}{28 \cdot 6^{3/2}} = 0.0131\ldots.$

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- Relying on the above result, we identify a critical line $(\nu, \tau(\nu))$ for $\nu > 1$, and a unique critical point (ν_c, t_c) on the critical line at which a *phase transition* occurs.
- $t_c(\nu) := \tau(\nu)$ is simply the radius of convergence of $z_{1,0}(t,\nu)$ for a fixed $\nu > 1$.

Theorem [Chen, T., 2020] (spins on faces) ([2])

For $\nu > 1$,

$$z_{p,q}(t_c(\nu),\nu) \sim \frac{a_p(\nu)}{\Gamma(-\alpha_0)} u_c(\nu)^{-q} \; q^{-(\alpha_0+1)} \qquad \text{as } q \to \infty;$$

$$a_p(\nu) \sim \frac{b(\nu)}{\Gamma(-\alpha_1)} u_c(\nu)^{-p} p^{-(\alpha_1+1)} \qquad \text{as } p \to \infty;$$

$$z_{p,q}(t_c(\nu),\nu) \sim \frac{b(\nu)\cdot c(q/p)}{\Gamma(-\alpha_0)\Gamma(-\alpha_1)} u_c(\nu)^{-(p+q)} p^{-(\alpha_2+2)} \quad \text{as } p,q \to \infty$$

$$\text{while } q/p \in [\lambda_{\min},\lambda_{\max}] \text{ where } 0 < \lambda_{\min} < \lambda_{\max} < \infty.$$

The perimeter exponents are determined by the following table:

$\nu \in$	$(1,\nu_c)$	$\{\nu_c\}$	(ν_c,∞)
α_0	$3/2$	$4/3$	$3/2$
α_1	-1	$1/3$	$3/2$
α_2	$1/2$	$5/3$	3

Proof ideas

We want to understand the singularity structure of $Z(u, v; \nu)$, which boils down to understanding the one of the RP $(\hat{Z}(H, K; S), \hat{U}(H; S), \hat{U}(K; S))$ with $\nu = \hat{\nu}(S)$. This involves:

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- Showing that $u_c(\nu)$ is the unique dominant singularity of Z, which in particular involves showing that $(H_c(S), H_c(S))$ is the only possible pole of $\hat{Z}$ which is mapped to $\partial D(0, u_c)^2$ under the aforementioned conformal bijection.

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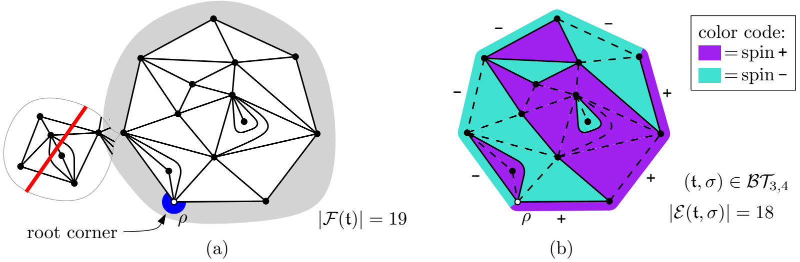
- Deducing that Z is holomorphic in a product of Δ-domains, which roughly means that it is amenable for transfer theorems of analytic combinatorics (see the book of Flajolet and Sedgewick).

A "geometric" reminder



Boltzmann distribution

Definition

The Boltzmann Ising-triangulation of the (p, q)-gon is a random variable having the law

$$\mathbb{P}^{\nu}_{p,q}(\mathfrak{t}, \sigma) = \frac{t_c(\nu)^{\mathrm{Vol}(\mathfrak{t})} \nu^{\mathcal{E}(\mathfrak{t},\sigma)}}{z_{p,q}(t_c(\nu), \nu)},$$

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In the previous example,
$|\mathcal{F}(\mathfrak{t})| = 19$,
$|\mathcal{E}(\mathfrak{t},\sigma)| = 18$ and

$$\mathbb{P}^{\nu}_{p,q}(\mathfrak{t},\sigma) = \frac{t_c(\nu)^{19}\nu^{18}}{z_{3,4}(t_c(\nu),\nu)}$$

Definition

The local distance between Ising triangulations $\mathfrak{t}$ and $\mathfrak{t}'$ is

$$d_{\mathrm{loc}}(\mathfrak{t},\mathfrak{t}') = 2^{-R} \quad \text{with} \quad R = \sup\{r \geq 0 | B_r(\mathfrak{t}) = B_r(\mathfrak{t}')\}$$

where $B_r(\mathfrak{t})$, the ball of radius r in $\mathfrak{t}$, is the bicolored rooted map formed by all the internal faces (triangles) of $\mathfrak{t}$ having at least one vertex at a distance strictly smaller than r from the root vertex. The condition $B_r(\mathfrak{t}) = B_r(\mathfrak{t}')$ requires the two rooted maps to have the same coloring on the internal faces and the same boundary conditions.

Local limits

Theorem [Chen, T., 2020] (spins on faces) ([2])

For every $\nu > 1$, there exist probability distributions $\mathbb{P}_p^\nu$ and $\mathbb{P}_\infty^\nu$ supported on infinite bicolored triangulations with a boundary such that

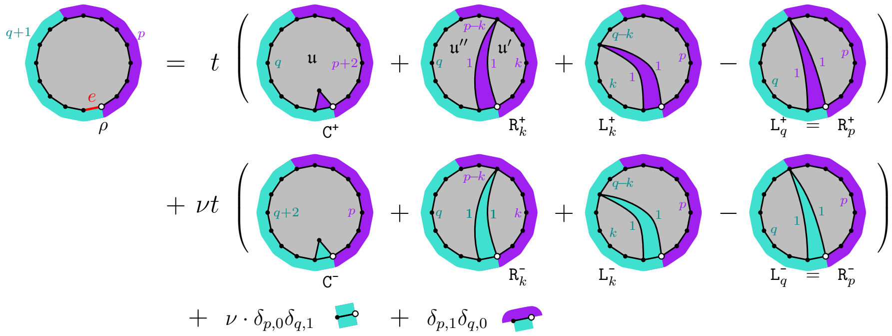
$$\mathbb{P}_{p,q}^\nu \xrightarrow[q\to\infty]{(d)} \mathbb{P}_p^\nu \xrightarrow[p\to\infty]{(d)} \mathbb{P}_\infty^\nu$$

in distribution w.r.t. the local distance. In addition, for $0 < \lambda' \leq 1 \leq \lambda < \infty$, we have

$$\mathbb{P}_{p,q}^\nu \xrightarrow[p,q\to\infty]{(d)} \mathbb{P}_\infty^\nu \qquad \text{while} \quad \frac{q}{p} \in [\lambda', \lambda]$$

locally in distribution. The local limits are one-ended, except $\mathbb{P}_\infty^\nu$ for $\nu > \nu_c$ which is two-ended.

Tutte's equation / peeling process



- The above recursive equation defines a probability distribution on $\mathcal{S} = \{C^+, C^-\} \cup \{L_k^+, L_k^-, R_k^+, R_k^- : k \geq 0\}$.
- Can be seen as the distribution of the first peeling step S_1 of a peeling process of $(\mathfrak{t}, \sigma) \sim \mathbb{P}_{p,q}^{\nu}$

Peeling process

- It is easy to verify that $\mathbb{P}^{\nu}_{p}(S) := \lim_{q\to\infty} \mathbb{P}^{\nu}_{p,q}(S)$ and $\mathbb{P}^{\nu}_{\infty}(S) := \lim_{p\to\infty} \mathbb{P}^{\nu}_{p}(S)$ also define probability distributions on $\mathcal{S}$.

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- Iterate the one-step peeling to obtain $\mathfrak{e}_n$ and $\mathfrak{u}_n$, the explored and unexplored maps after n peeling steps, respectively (so that $\mathfrak{e}_0$ just consists of the boundary of $(\mathfrak{t}, \sigma)$).

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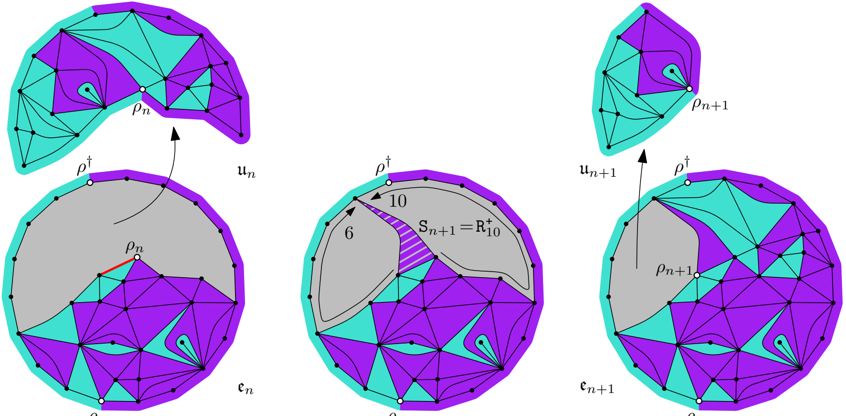
- Iterate the one-step peeling to obtain $\mathfrak{e}_n$ and $\mathfrak{u}_n$, the explored and unexplored maps after n peeling steps, respectively (so that $\mathfrak{e}_0$ just consists of the boundary of $(\mathfrak{t}, \sigma)$).
- To each unexplored map $\mathfrak{u}_n$, associate the perimeter (P_n, Q_n), giving rise to the *perimeter processes*.
- For $p, q < \infty$, define the *perimeter variation process* $(X_n, Y_n) = (P_n - p, Q_n - q)$, which can also be defined for $p, q = \infty$ and turns out to be a random walk under $\mathbb{P}^{\nu}_{\infty}$.

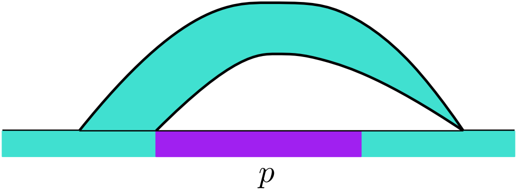


(a) $\mathfrak{e}_n$ and $\mathfrak{u}_n$ (b) The peeling step $\mathtt{S}_{n+1}$ (c) $\mathfrak{e}_{n+1}$ and $\mathfrak{u}_{n+1}$

The phase transition can be also seen from the following simple property, which defines an order parameter:

$$\mathbb{P}_p(\text{"The finite boundary is swallowed in a single step"})$$

$$\sim_{p\to\infty} \begin{cases} c(\nu)p^{-5/2} & (\nu < \nu_c) \\ cp^{-1} & (\nu = \nu_c) \\ c(\nu) \in (0,1) & (\nu > \nu_c) \end{cases} .$$



A glimpse of geometry; a phase transition

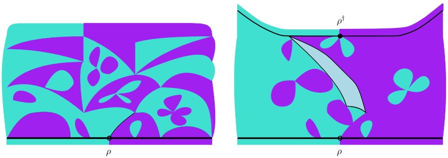


Figure: The local limits ($p, q \to \infty$) in the high temperature and the low temperature regimes.

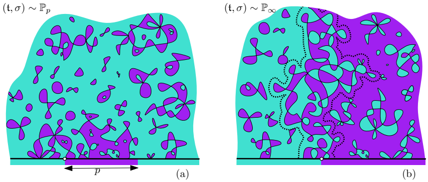


Figure: The two local limits at the critical temperature.

Interfaces at the critical temperature

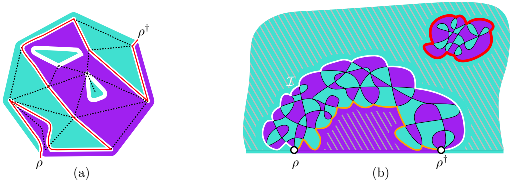


Figure: Spin cluster interfaces when the spins are on faces.

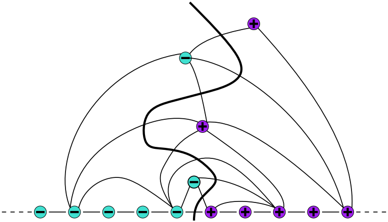


Figure: The unique infinite interface when the spins are on vertices.

Theorem [T., 2020] (spins on vertices) ([3])

Let $\nu = \nu_c$. Then the interface lengths $\eta_{p,q}$ and η_p between the marked boundary vertices ρ and $\rho^\dagger$ (at the junctions of the + and − boundaries) have the following scaling limits:

$$\forall t > 0\,, \qquad \lim_{p,q\to\infty} \mathrm{P}_{p,q}\left(\mu\eta_{p,q} > tp\right) = \int_t^\infty (1+s)^{-7/3}(\lambda+s)^{-7/3} ds$$

where μ is an explicit constant and the limit is taken such that $q/p \to \lambda \in (0,\infty)$. In particular, for $\lambda = 1$,

$$\lim_{p,q\to\infty} \mathrm{P}_{p,q}\left(\eta_{p,q} > tp\right) = (1+\mu t)^{-11/3}.$$

Moreover,

$$\forall t > 0\,, \qquad \lim_{p\to\infty} \mathrm{P}_{p}\left(\eta_{p} > tp\right) = (1+\mu t)^{-4/3}.$$

Interpretation

- Consider two independent $\sqrt{3}$-LQG *quantum disks* with two marked points and perimeters $1 + L$ and $L + \lambda$, respectively, and their conformal welding, as defined in [Ang, Holden, Sun [3]].

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- Similarly, the law $\mathrm{P}(\tilde{L} > t) := (1 + \mu t)^{-4/3}$ is the law of the interface length in the conformal welding of a quantum disk with a *thick quantum wedge*.

A closely related work

Albenque, Ménard and Schaeffer [2] considered the set of triangulations of the *sphere* of size n decorated with Ising model on the *vertices*.

- It is shown that for *any* fixed $\nu > 0$, the law $\mathbb{P}_n$ of the random triangulation converges weakly in the local topology when $n \to \infty$.
- The above local limit at the critical temperature is shown to be a.s. recurrent.

Works in progress and future directions

- Near-critical regime
- Universality
- More general boundary conditions; crossing probabilities
- Scaling limits of perimeter and volume
- Scaling limits of the interface with conformal structure ($\to$ LQG+SLE), ..., full scaling limit of the model as a LQG surface?

This talk was based on:

- L. Chen and J. Turunen. Critical Ising model on random triangulations of the disk: enumeration and local limits. *Commun. Math. Phys.*374, 1577–1643 (2020).
- L. Chen and J. Turunen. Ising model on random triangulations of the disk: phase transition. *arXiv:2003.09343*, 2020.
- J. Turunen. Interfaces in the vertex-decorated Ising model on random triangulations of the disk. *arXiv:2003.11012*, 2020.

Related works:

- O. Bernardi and M. Bousquet-Melou. Counting colored planar maps: algebraicity results. *J. Combin. Theory Ser. B*, 101(5):315-377, 2011.
- M. Albenque and L. Ménard and G. Schaeffer. Local convergence of large random triangulations coupled with an Ising model. *Trans. Amer. Math. Soc.* (accepted), 2020.
- M. Ang and N. Holden and X. Sun. Conformal welding of quantum disks. *arXiv:2009.08389*, 2020.

Merci beaucoup!