Tau-functions à la Dubédat and cylindrical events in the double-dimer model

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joint work w/ Dmitry Chelkak

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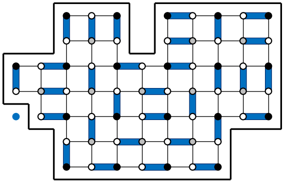
Setup: double-dimer loop ensembles in Temperley discretizations on $\mathbb{Z}^2$

• **Temperley discretizations Ω^δ on $\mathbb{Z}^2$:** simply connected domains s.t. all corners are of the same type out of four: B_0, B_1, W_0, W_1.

• **Dimer ($=$ domino) model on Ω^δ:** perfect matchings, chosen uniformly at random.



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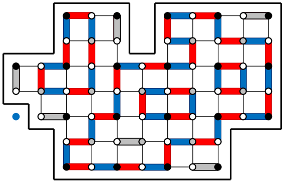

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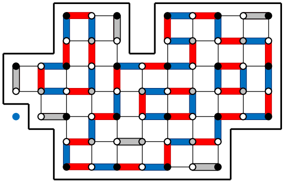
• **Double-dimer model:** two independent dimer configurations on the same domain. Configuration $\mathcal{L}^{\text{dbl-d}}$ is a fully-packed collection of loops and double-edges,

$$\mathcal{Z}^{\text{dbl-d}} = \sum_{\mathcal{L}^{\text{dbl-d}}} 2^{\#\text{loops}(\mathcal{L}^{\text{dbl-d}})}$$

Big Goal (cf. Kenyon'10, Dubédat'14): conformal invariance, convergence to CLE_4

• **Rand. height function vs GFF:** Choosing the orientation of loops $\gamma \in \mathcal{L}^{\text{dbl-d}}$ randomly, one gets a height function $h^{\text{dbl-d}}$.



Kenyon'00: $h^{\text{dbl-d}} \to \mathrm{GFF}(\Omega)$ as $\delta \to 0$ (in Temperley bc).

[!] The convergence of $h^{\text{dbl-d}}$ is not strong enough for the level lines $\mathcal{L}^{\text{dbl-d}}$ of $h^{\text{dbl-d}}$.

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Concrete goal we discuss:

convergence of probabilities $\boldsymbol{p}_\Gamma^\delta$ of cylindrical events

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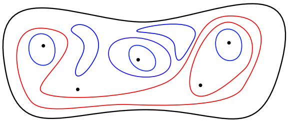
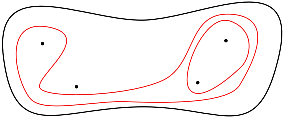


- Lamination Γ = collection of loops in $\Omega \setminus \{\lambda_1, \ldots, \lambda_n\}$ up to homotopies

- Γ macroscopic $\leftrightsquigarrow$ any $\gamma \in \Gamma$ has ≥ 2 punctures inside



- remove loops with ≤ 1 punctures inside from $\mathcal{L}$. The rest $=: \mathcal{L}_{\mathrm{macro}}$

- cylindrical event for Γ := set of all $\mathcal{L}$ s.t. $\mathcal{L}_{\mathrm{macro}} \simeq \Gamma$.

$$\boldsymbol{p}_\Gamma^\delta := 2^{-\#\mathrm{loops}(\Gamma)} \cdot \mathbb{P}[\boldsymbol{\mathcal{L}}_{\mathbf{macro}}^{\mathbf{dbl\text{-}d}} \simeq \boldsymbol{\Gamma}]$$

Concrete goal we discuss:

Theorem: for each macroscopic Γ we have $\lim_{\delta \to 0} p_\Gamma^\delta = p_\Gamma^{\mathrm{CLE}_4}$.

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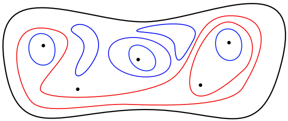

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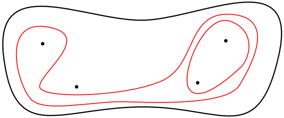

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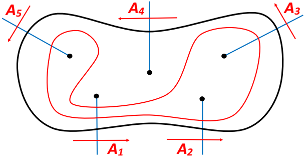
Kenyon (2010): $\mathrm{SL}_2(\mathbb{C})$-monodromies and Q-determinants for double-dimers

$\rho : \pi_1(\Omega \setminus \{\lambda_1, \ldots, \lambda_n\}) \to \mathrm{SL}_2(\mathbb{C})$.

Down-to-earth viewpoint: draw cuts from punctures λ_k to $\partial\Omega$ and choose $A_k \in \mathrm{SL}_2(\mathbb{C})$.



$$\mathbb{E}\Big[\prod_{\gamma \in \mathcal{L}^{\mathrm{dbl\text{-}d}}} \big(\tfrac{1}{2}\,\mathrm{Tr}\,\rho(\gamma)\big)\Big] = \frac{\det \mathcal{K}^{(\rho)}}{(\det \mathcal{K})^2}\,, \qquad \rho(\gamma) = A_5 A_1^{-1} A_3 A_2 A_1$$

where $\mathcal{K}^{(\rho)} : (\mathbb{C}^2)^{\mathcal{B}} \to (\mathbb{C}^2)^{\mathcal{W}}$ is obtained from $\mathcal{K}$ by putting the matrices $A_k^{\pm 1}$ on cuts.

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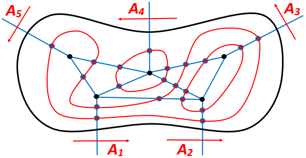

$n(L)=(2,2,2,1,1,1,2,0,1,3,3,1,2)_{e\in\mathcal{E}}$

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Remark: A better viewpoint is to fix a triangulation of $\Omega \setminus \{\lambda_1, \ldots, \lambda_n\}$ and to consider discrete $\mathbb{C}^2$-vector bundles and flat $\mathrm{SL}_2(\mathbb{C})$-connections on them:

$$\frac{\{\rho : \pi_1(\Omega \setminus \{\lambda_1, \ldots, \lambda_n\}) \to \mathrm{SL}_2(\mathbb{C})\}}{\mathrm{SL}_2(\mathbb{C})} \simeq \frac{\mathrm{SL}_2(\mathbb{C})^{\mathcal{E}}}{\mathrm{SL}_2(\mathbb{C})^{\mathcal{F}}}.$$

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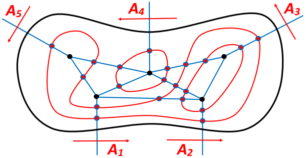

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where $\mathcal{K}^{(\rho)} : (\mathbb{C}^2)^{\mathcal{B}} \to (\mathbb{C}^2)^{\mathcal{W}}$ is obtained from $\mathcal{K}$ by putting the matrices $A_k^{\pm 1}$ on cuts.

Representation $\rho : \pi_1(\Omega \setminus \{\lambda_1, \dots, \lambda_n\}) \to \mathrm{SL}_2(\mathbb{C})$ is unipotent iff $\mathbf{Tr}\,\boldsymbol{\rho}([\boldsymbol{\lambda_k}]) = 2$ for each of the loops $[\lambda_k]$ surrounding a single puncture λ_k.

Dubédat (2014): locally unipotent monodromies and convergence to the isomonodronic τ-function

$\rho : \pi_1(\Omega \setminus \{\lambda_1, \ldots, \lambda_n\}) \to \mathrm{SL}_2(\mathbb{C})$ unipotent iff $\mathbf{Tr}\, \boldsymbol{\rho}([\boldsymbol{\lambda_k}]) = 2$ for each of the loops $[\lambda_k]$ surrounding a single puncture λ_k.

Let Ω^δ, $\delta \to 0$, be a sequence of Temperley approximations to a simply connected domain $\Omega \subset \mathbb{C}$. Fix punctures $\lambda_1, \ldots, \lambda_n \in \Omega$.

Theorem (Dubédat, 2014): Let $\rho : \pi_1(\Omega \setminus \{\lambda_1, \ldots, \lambda_n\}) \to \mathrm{SL}_2(\mathbb{C})$ be unipotent. Then

$$\mathbb{E}\big[\prod_{\gamma \in \mathcal{L}^{\text{dbl-d}}} \big(\tfrac{1}{2}\,\mathrm{Tr}\,\rho(\gamma)\big)\big] \;=:\; \boldsymbol{\tau^{\delta}(\rho)} \;\to\; \boldsymbol{\tau^{\mathrm{iso}}(\rho)} \quad \text{as} \quad \delta \to 0.$$

Remark: In fact, the convergence is uniform on compacts of

$$\boldsymbol{X}_{\mathrm{unip}} \subset \boldsymbol{X} := \{\rho : \pi_1(\Omega \setminus \{\lambda_1, \ldots, \lambda_n\}) \to \mathrm{SL}_2(\mathbb{C})\}.$$

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We have

$$\boldsymbol{\tau}^{\boldsymbol{\delta}}(\boldsymbol{\rho}) = \mathbb{E}\big[\prod_{\gamma\in\mathcal{L}^{\text{dbl-d}}}\big(\tfrac{1}{2}\operatorname{Tr}\rho(\gamma)\big)\big] \;=\sum_{\Gamma-\text{macro}} p_\Gamma^\delta\cdot f_\Gamma(\rho),$$

where

$$f_\Gamma(\rho) := \prod_{\gamma\in\Gamma}\operatorname{Tr}\rho(\gamma), \qquad \{f_\Gamma\}_\Gamma \text{ — basis in } \mathrm{Fun}(\mathrm{X})^{\mathrm{SL}_2(\mathbb{C})}$$

Convergence of $\boldsymbol{p}_\Gamma^{\boldsymbol{\delta}}$ is thus reduced to a statement of the form "convergence of *infinite* series" ⇒ "convergence of coefficients"

Theorem (Dubédat, 2014): Let $\rho : \pi_1(\Omega\setminus\{\lambda_1,\dots,\lambda_n\})\to \mathrm{SL}_2(\mathbb{C})$ be unipotent. Then

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$$\boldsymbol{X}_{\mathrm{unip}}\subset \boldsymbol{X} := \{\rho : \pi_1(\Omega\setminus\{\lambda_1,\dots,\lambda_n\})\to\mathrm{SL}_2(\mathbb{C})\}.$$

Main result (joint w/ Dmitry Chelkak, 2018)

Main message: $F : X_{\mathrm{unip}} \to \mathbb{C}$, holomorphic, $\mathrm{SL}_2(\mathbb{C})$ - inv., then

$$F(\rho) \;=\; \sum\nolimits_{\Gamma-\mathrm{macro}} p_\Gamma f_\Gamma(\rho).$$

Theorem: There exists an absolute constant $k_0 > 1$ such that:

(i) if $R > R_0$ and $F : X_{\mathrm{unip}} \cap \overline{\mathbb{D}}_R \to \mathbb{C}$ is holomorphic, then there exist coefficients $p_\Gamma = O\big(\big(\frac{R}{R_0}\big)^{-|n(\Gamma)|} \cdot \|F\|_{L^\infty(\overline{\mathbb{D}}_R)}\big)$ s.t.

$$F(\rho) \;=\; \sum\nolimits_{\Gamma-\mathrm{macro}} p_\Gamma f_\Gamma(\rho), \quad \rho \in X_{\mathrm{unip}} \cap \mathbb{D}_{RR_0^{-1}}.$$

(ii) if $r > R_0$ and coefficients $p_\Gamma, \tilde{p}_\Gamma = O(r^{-|n(\Gamma)|})$ satisfy

$$\sum\nolimits_{\Gamma-\mathrm{macro}} p_\Gamma f_\Gamma(\rho) \;=\; \sum\nolimits_{\Gamma-\mathrm{macro}} \tilde{p}_\Gamma f_\Gamma(\rho), \quad \rho \in X_{\mathrm{unip}} \cap \mathbb{D}_r.$$

Then $p_\Gamma = \tilde{p}_\Gamma$ for all macroscopic laminations Γ.

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Main result (joint w/ Dmitry Chelkak, 2018)

Corollary: τ^{iso} is holomoprhic X_{unip} $\Rightarrow$ there exist unique coefficients p_Γ^{iso} s.t. $\tau^{\text{iso}}(\rho) = \sum_{\Gamma-\text{macro}} p_\Gamma^{\text{iso}} f_\Gamma(\rho)$, $\rho \in X_{\text{unip}}$.

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Corollary: $\boldsymbol{\tau}^{\mathbf{iso}}$ is holomoprhic $X_{\mathrm{unip}} \Rightarrow$ there exist unique coefficients p_Γ^{iso} s.t. $\tau^{\mathrm{iso}}(\rho) = \sum_{\Gamma-\mathrm{macro}} p_\Gamma^{\mathrm{iso}} f_\Gamma(\rho)$, $\rho \in X_{\mathrm{unip}}$.

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$$F(\rho) \ = \ \sum\nolimits_{\Gamma-\mathrm{macro}} p_\Gamma f_\Gamma(\rho), \quad \rho \in X_{\mathrm{unip}} \cap \mathbb{D}_{RR_0^{-1}}\,.$$

Corollary: (a) Uniform boundedness of τ^δ on $\overline{\mathbb{D}}_R$ for any $R > 0$ implies the uniform (in δ) estimate $p_\Gamma^\delta = O(r^{-|n(\Gamma)|})$ for all $r > 0$.

(b) Convergence (as $\delta \to 0$) of $\tau^\delta \to \tau^{\mathrm{iso}}$ on $\overline{\mathbb{D}}_R$ implies $p_\Gamma^\delta \to p_\Gamma^{\mathrm{iso}}$ for all macroscopic laminations Γ.

Relation to CLE_4

Corollary: $\boldsymbol{\tau}^{\mathbf{iso}}$ is holomoprhic $X_{\mathrm{unip}} \Rightarrow$ there exist unique coefficients p_Γ^{iso} s.t. $\tau^{\mathrm{iso}}(\rho) = \sum_{\Gamma - \mathrm{macro}} p_\Gamma^{\mathrm{iso}} f_\Gamma(\rho)$, $\rho \in X_{\mathrm{unip}}$.

Theorem (Dubédat): If $\rho \in X_{\mathrm{unip}}$ is close enough to Id, then

$$\boldsymbol{\tau}^{\mathbf{iso}}(\boldsymbol{\rho}) \;=\; \mathbb{E}\big[\prod_{\gamma \in \mathcal{L}^{\mathrm{CLE}_4}} (\tfrac{1}{2}\,\mathrm{Tr}\,\rho(\gamma))\big] \;=\; \sum_{\Gamma - \mathrm{macro}} p_\Gamma^{\mathrm{CLE}_4} f_\Gamma(\rho)$$

(in particular, the r.h.s. converges)

Remark: this does not directly imply $p_\Gamma^{\mathrm{CLE}_4} = p_\Gamma^{\mathrm{iso}}$:
a superexponential decay of $p_\Gamma^{\mathrm{CLE}_4}$ needed.

Relation to CLE_4

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Theorem(Bai, Wan): for any $R>0$, $p_\Gamma^{\mathrm{CLE}_4} = O(R^{-|n(\Gamma)|})$

Conclusions:

(i) For any macroscopic Γ, $p_\Gamma^{\mathrm{iso}} = p_\Gamma^{\mathrm{CLE}_4}$, therefore $\lim_{\delta\to 0} p_\Gamma^\delta = p_\Gamma^{\mathrm{CLE}_4}$

(ii) To claim the

convergence of double-dimes loop ensembles to CLE_4

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Thank you for your attention!