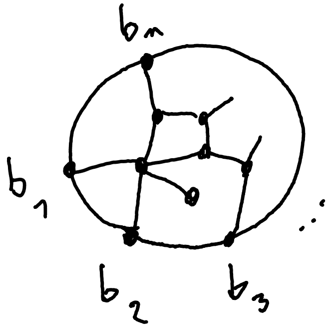
Notes: Boundary correlations for the Z-invariant Ising Model

References:
- Ising Model and the positive orthogonal Grassmanian, P. Galashin and P. Pylyavskyy 2020
- A formula for boundary correlations of the critical Ising model, P. Galashin 2020

I The Ising Model



G planar graph embedded in a disk with n boundary vertices

$J : E(G) \longrightarrow \mathbb{R}_+$ weights on the edges

For $\sigma \in \{-1,1\}^{|V|}$ $\quad \mathbb{P}(\sigma) := \frac{1}{Z} \prod_{\{u,v\}\in E} \exp\left(J_{\{u,v\}}\sigma_u\sigma_v\right)$

We are interested in boundary correlations:

$\sigma_{ij} := \mathbb{E}(\sigma_{b_i}\sigma_{b_j})$

$M := (\sigma_{ij})_{1\leq i,j\leq n}$ boundary correlations matrix

From this model, there are two ways to assign an element of $OG_{\geq 0}(n, 2n)$ (defined later):

1) One from the matrix M
2) One using dimers

These two ways will in fact provide the same element of $OG_{\geq 0}(n, 2n)$.

About 1)

$$M = (\sigma_{ij}) \longrightarrow \tilde{M} = (\tilde{m}_{ij}) \text{ defined by}$$

$$\forall i \quad \tilde{m}_{i,2i-1} = \tilde{m}_{i,2i} = 1$$

$$\forall i \neq j \quad \tilde{m}_{i,2j-1} = -\tilde{m}_{i,2j} = (-1)^{i+j+\mathbb{1}_{i<j}} m_{ij}$$

Ex: $M = \begin{pmatrix} 1 & \sigma_{12} \\ \sigma_{12} & 1 \end{pmatrix}$

$$\longrightarrow \tilde{M} = \begin{pmatrix} 1 & 1 & \sigma_{12} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{12} & 1 & 1 \end{pmatrix}$$

For I a subset of size n of $[2n]$ define

$\Delta_I(\tilde{M}) :=$ principal minor with columns indexed by I.

In our example,

$$\Delta_{\{1,2\}}(\tilde{M}) = \begin{vmatrix} 1 & 1 \\ -\sigma_{12} & \sigma_{12} \end{vmatrix} = 2\sigma_{12} \geq 0$$

$$\Delta_{\{3,4\}}(\tilde{M}) = \begin{vmatrix} \sigma_{12} & -\sigma_{12} \\ 1 & 1 \end{vmatrix} = 2\sigma_{12} = \Delta_{\{1,2\}}(\tilde{M})$$

In general, $\Delta_I(\tilde{M}) \geq 0 \quad \forall I$

$$\Delta_I(\tilde{M}) = \Delta_{[2n]\setminus I}(\tilde{M}) \quad \forall I.$$

II The Positive Orthogonal Grassmannian

$Gr(n, 2n) = \{$subspace of size n of $\mathbb{R}^{2n}\}$

- An element of $Gr(n,2n)$ can be represented by a full rank $n \times 2n$ matrix
- If two matrices M and M' represent the same element in $Gr(n,2n)$ then $\exists g$ invertible such that $M = g \cdot M'$

So $\Delta_I(M) = \det(g)\Delta_I(M')$ $\forall I$

Therefore for $X \in Gr(n, 2n)$, we can define $(\Delta_I(X))_{I \in \binom{[2n]}{[n]}}$ up to a multiplicative constant.

- $Gr_{\geq 0}(n, 2n) := \{ X \in Gr(n, 2n) : \Delta_I(X) \geq 0 \ \forall I \}$

$OG(n, 2n) := \{ X \in Gr(n, 2n) : \Delta_I(X) = \Delta_{I \setminus [2n]}(X) \ \forall I \}$

$OG_{\geq 0}(n, 2n) = Gr_{\geq 0}(n, 2n) \cap OG(n, 2n)$

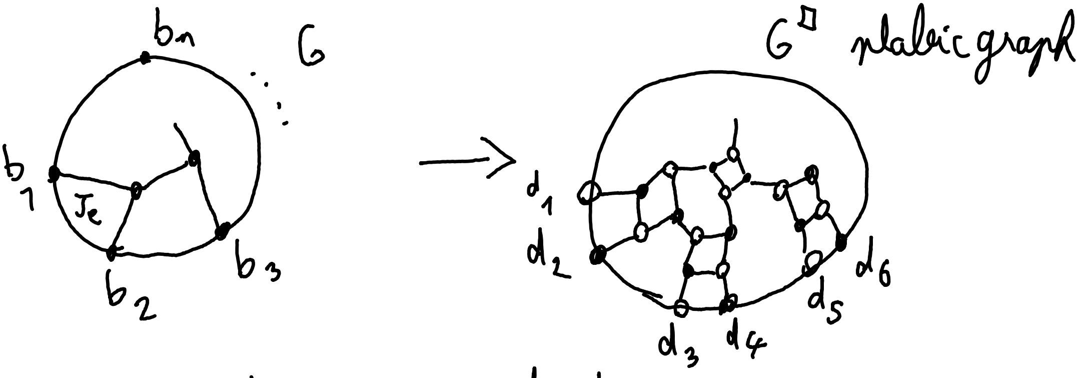
- For $M = (6_{ij})$ we have $\tilde{M}$ representing an element of $OG_{\geq 0}(n, 2n)$

Furthermore, one can prove that

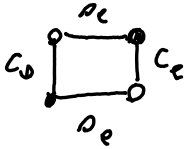
$$6_{ij} = \frac{\sum_{I \in \mathcal{E}_n(\{i,j\})} \Delta_I(\tilde{M})}{\sum_{I \in \mathcal{E}_n(\emptyset)} \Delta_I(\tilde{M})}$$

where for $S \subset [n]$, $\mathcal{E}_n(S) = \{ I \in \binom{[2n]}{[n]}$ such that $\forall i$ $I \cap \{2i-1, 2i\}$ has even size if and only if $i \in S \}$

III Correspondance with dimers



- To each $\overset{b_j}{\bullet} \longrightarrow \overset{d_\ell}{\circ}\ \ \overset{d_{\ell+1}}{\bullet}$
- To each edge $\underset{J_e}{\overline{\qquad}} \longrightarrow$

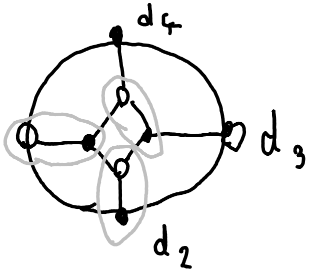


$$s_e := (\cosh(2J_e))^{-1}$$
$$c_e := \tanh(2J_e)$$

- link the squares if the edges were incident.
- Other edges have weight 1.

Def: • An almost dimer configuration is a set of edges such that each interior vertex is incident to exactly one edge in the set (no constraint for boundary vertices)

- $\partial\mathcal{A} = \{$black boundary vertices used in $\mathcal{A}\}$
  $\cup\ \{$white boundary vertices not used in $\mathcal{A}\}$

Ex: 

$\partial \mathcal{A} = \{2,3\}$

Lem: $|\partial \mathcal{A}| = n$

We can define a collection of numbers $(\Delta_I(G^\square))_{I \in \binom{[2n]}{[n]}}$ by

$$\Delta_I(G^\square) := \sum_{\mathcal{A} : \partial \mathcal{A} = I} w(\mathcal{A})$$

weight of the almost dimer config

- We clearly have $\forall I$ $\Delta_I(G^\square) \geq 0$
- One can show that $\forall I$ $\Delta_I(G^\square) = \Delta_{[2n] \setminus I}(G^\square)$

To make sure that these numbers can be seen as principal minors of a matrix, one should also check that they satisfy a quadratic relation called the Plücker relation.

$\longrightarrow \exists! X' \in OG_{\geq 0}(n, 2n)$ s.t. $\Delta_I(X') = \Delta_I(G^\square)$ $\forall I$

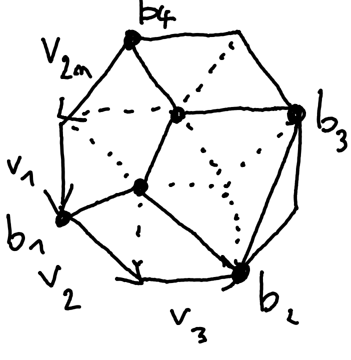
Thm: $$\underset{\substack{\text{obtained}\\ \text{from } \hat{\pi}}}{X} = \underset{\substack{\text{obtained}\\ \text{from } G^{\square}}}{X'}$$

$$\Rightarrow \quad \sigma_{ij} = \frac{\displaystyle\sum_{\mathcal{A} : \partial\mathcal{A} \in \mathcal{E}_n(\{i,j\})} w(\mathcal{A})}{\displaystyle\sum_{\mathcal{A} : \partial\mathcal{A} \in \mathcal{E}_n(\emptyset)} w(\mathcal{A})}$$

IV Practical use

We now consider the following graph:

R region tileable with rhombi made of $2n$ unit vectors (for instance, R is a regular polygon)

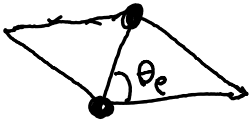


Let T be a rhombus tiling. Define G^T the graph obtained by having a vertex between v_{2j-1} and v_{2j} and by linking the diagonals.

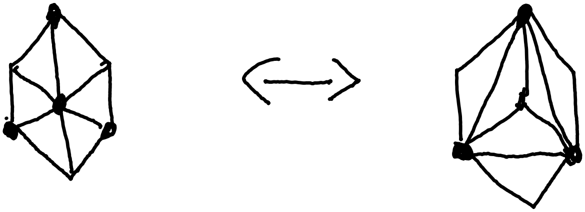
Define $J : E \to \mathbb{R}_+$

$$e \longrightarrow J_e := \frac{1}{2} \ln \left(\frac{1 + \mathrm{sn}\left(\frac{2K(k)}{\pi} \theta_e \,|\, k\right)}{\mathrm{cn}\left(\frac{2K(k)}{\pi} \theta_e \,|\, k\right)} \right)$$

where sn, cn are Jacobi elliptic functions, k is the elliptic parameter and θ_e is the semi-angle defined by:



these weights satisfy a Yang-Baxter equation which means in our context that boundary correlation are not changed by flipping a cube:

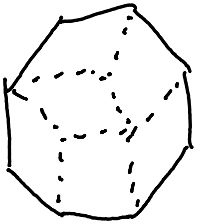


$M = (\sigma_{ij})_{1 \leq i,j \leq n}$ depends only on the region R and not on the tiling.

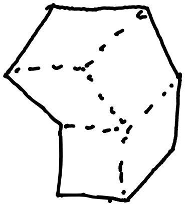
- Write $v_j = e^{i 2\alpha_j}$

  $\underline{\alpha} = (\alpha_1, \dots, \alpha_{2n})$

Prop: Let R be a region ($\underline{\alpha}$ the angles associated to it) Let R′ be the region obtained by removing a rhombus with two sides v_j and v_{j+1} on the border.



R



R′

Then $\mathrm{rowspan}(\tilde{M}_R) = \mathrm{rowspan}(\hat{M}_{R'}) \cdot g_j^{\underline{\alpha}}$

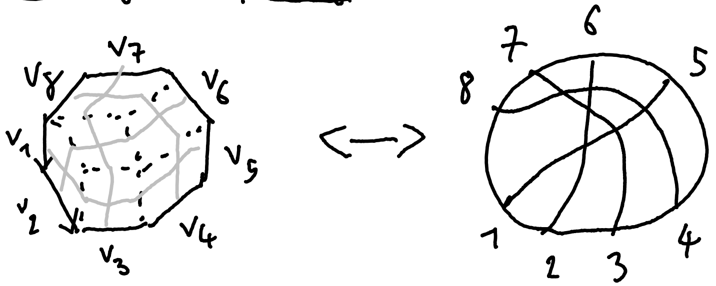
where $g_j^{\underline{\alpha}} = \begin{pmatrix} I & & 0 \\ & B_j & \\ 0 & & I \end{pmatrix}$

with $B_j = \begin{cases} \begin{pmatrix} \frac{1}{cn} & \frac{sn}{cn} \\ \frac{sn}{cn} & \frac{1}{cn} \end{pmatrix} & \text{if } j \text{ even} \\ \begin{pmatrix} \frac{1}{cn^*} & \frac{sn^*}{cn^*} \\ \frac{sn^*}{cn^*} & \frac{1}{cn^*} \end{pmatrix} & \text{if } j \text{ odd} \end{cases}$

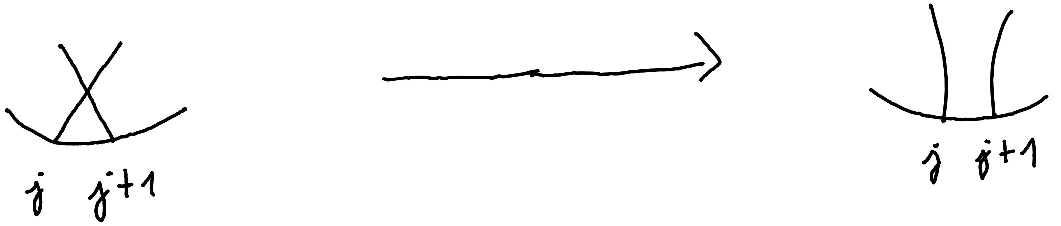
where $$\mathrm{sn} := \mathrm{sn}\left(\frac{2K(k)}{\pi}(\alpha_{j+1}-\alpha_j)\,\middle|\,k\right)$$

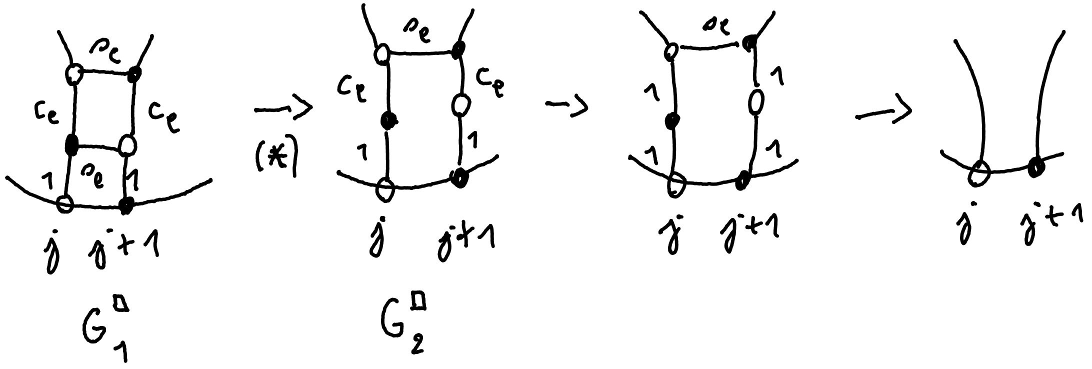
$$\mathrm{sn}^* := \mathrm{sn}\left(\frac{2K(k^*)}{\pi}(\alpha_{j+1}-\alpha_j)\,\middle|\,\underset{\text{dual parameter}}{k^*}\right)$$

Ideas of the proof:



We want to understand





Study of step $(*)$

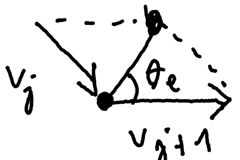
- $$\Delta_I(G_1^\square) = \begin{cases} \Delta_I(G_2^\square) & \text{if } j \notin I \text{ or } j+1 \in I \\ \Delta_I(G_2^\square) + s_e \, \Delta_{(I \setminus \{j+1\}) \cup \{j\}}(G_2^\square) & \text{o.w.} \end{cases}$$

It can be written in terms of matrices:

$$X(G_1^\square) = X(G_2^\square) \cdot x_j(s_e)$$

where $$x_j(s_e) = \begin{array}{cc} j & \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & s_e & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \\ & \begin{array}{ccccccc} \; & \; & \; & \; & j+1 & \; & \; \end{array} \end{array}$$

- For j odd,



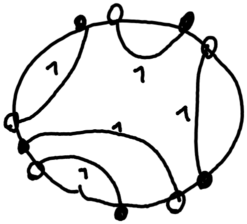
$$\theta_e = \frac{\pi}{2} - (\alpha_{j+1} - \alpha_j)$$

$$\begin{aligned} s_e &= (\cosh(2J_e))^{-1} \\ &= \operatorname{cn}\left(\frac{2K(k)}{\pi}\theta_e \,\middle|\, k\right) \\ &= \operatorname{sn}\left(\frac{2K(k^*)}{\pi}(\alpha_{j+1} - \alpha_j) \,\middle|\, k\right) \end{aligned}$$

$\square$

So we know how to go from k crossings to (k-1) crossings.

For 0 crossing :



$$\Delta_I(G^D_{\text{no crossing}}) = \begin{cases} 0 & \text{if } \{i, \overset{\text{vertex linked to } i}{\tau(i)}\} \subset I \text{ for a } i \in [2n] \\ 1 & \text{o.w.} \end{cases}$$

→ In theory, we can get a matrix A of size $n \times 2n$ such that $\mathrm{rowspan}(A) = \mathrm{rowspan}(\tilde{M}_R)$.

<u>Prop</u> : If A is such that $\mathrm{rowspan}(A) = \mathrm{rowspan}(\tilde{M}_R)$

Then $(AK_n)^{-1}A = \tilde{M}_R$

where $K_n := \frac{1}{2}\begin{pmatrix} 1 & & \\ 1 & & 0 \\ & 1 & \\ & 1 & \\ 0 & \ddots & 1 \\ & & 1 \end{pmatrix}$

<u>Proof</u> : It comes from the fact that $\tilde{M}_R K_n = I_n$

Using these ideas, P. Galashin proved the following result in the critical case.

Thm: Let R_n be a regular $2n$-polygon.
Suppose the elliptic parameter $k = 0$.
Then $\forall\, p, q \in [n]$

$$\sigma_{pq} = \frac{2}{n}\left(\frac{1}{\sin\left(\frac{(2k-1)\pi}{2n}\right)} - \frac{1}{\sin\left(\frac{(2k-3)\pi}{2n}\right)} + \dots \right.$$

$$\left. \dots \pm \frac{1}{\sin\left(\frac{\pi}{2n}\right)} \right) \mp 1$$