Limit shapes in quantum integrable spin chains

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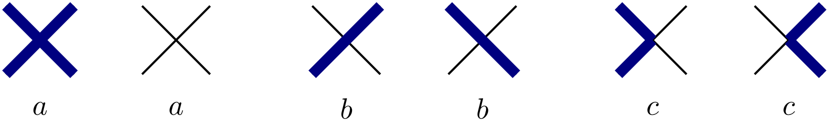
Dimers day, Saclay, 2022

[JMS, 2112.12092] [Bocini & JMS, 2007.06621] [JMS, 1707.06625]

Outline

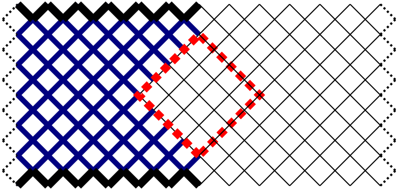
Six vertex model



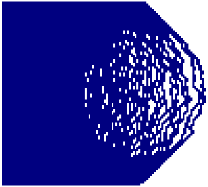
$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}$$

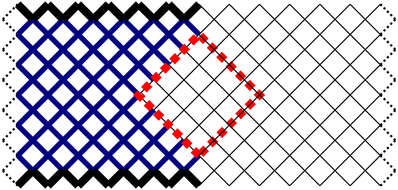
In this whole talk, $a = 1$, Δ fixed to some value $\in \mathbb{R}$.

Setup studied here, in pictures.



$$b = 1$$





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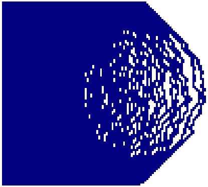
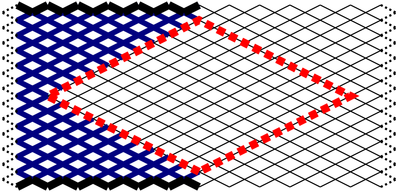
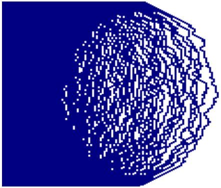
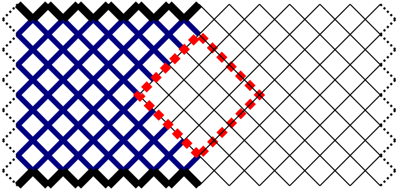




$b = \frac{1}{2}$





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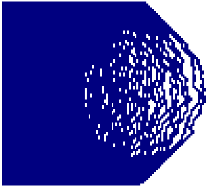

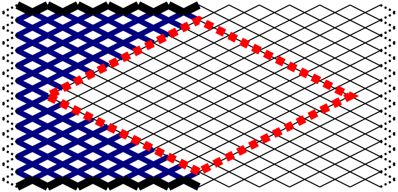

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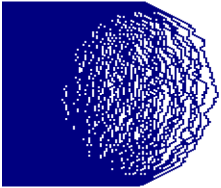
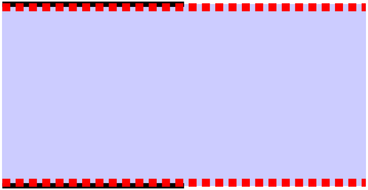
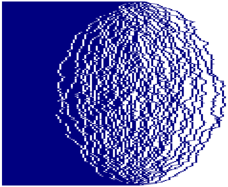




$b \to 0$



Spins and fermions

N is integer ≥ 2. Hilbert space with an orthonormal basis labelled by binary words of length N. Label sites $j \in \{1, \ldots, N\}$.

● $= 1 = +$ is a particle, ○ $= 0 = -$ is a hole.

There are 2^N basis states, one of which is shown below ($N = 8$):

|○○●●○●○○⟩

The allowed states $|v\rangle$ (column vectors) are linear combinations of basis states with complex coefficients.

$\langle v| := (|v\rangle)^H = (|v\rangle)^\dagger$ is corresponding line vector.

$\langle u|v\rangle := \langle u|\,|v\rangle$ scalar product between the vectors $|u\rangle$ and $|v\rangle$.

Spin flip operators

- $\sigma_i^- \sigma_j^+$: if there is a particle at site i and a hole at site $j \neq i$, moves the particle from i to j. Otherwise returns the 0 vector.
- $\sigma_i^+ \sigma_j^-$: if there is a particle at site j and a hole at site $i \neq j$, moves the particle from j to i. Otherwise returns the 0 vector.

$$\sigma_3^- \sigma_4^+ \left| \circ \circ \bullet \bullet \circ \bullet \circ \circ \right\rangle = 0$$

$$\sigma_4^- \sigma_5^+ \left| \circ \circ \bullet \bullet \circ \bullet \circ \circ \right\rangle = \left| \circ \circ \bullet \circ \bullet \bullet \circ \circ \right\rangle$$

$$\sigma_2^+ \sigma_3^- \left| \circ \circ \bullet \bullet \circ \bullet \circ \circ \right\rangle = \left| \circ \bullet \circ \bullet \circ \bullet \circ \circ \right\rangle$$

$$\sigma_3^+ \sigma_4^- \left| \circ \circ \bullet \bullet \circ \bullet \circ \circ \right\rangle = 0$$

Counting the interfaces between particles and holes

$$D\,|\circ\,\circ\vdots\bullet\,\bullet\vdots\circ\vdots\bullet\vdots\circ\,\circ\rangle = 4\,|\circ\,\circ\,\bullet\,\bullet\,\circ\,\bullet\,\circ\,\circ\rangle$$

D is a diagonal $2^N \times 2^N$ matrix.

Can also make local version d_j (also known as $\frac{1-\sigma_j^z\sigma_{j+1}^z}{2}$) counting whether there is one interface beween site j and site $j+1$. Then

$$D = \sum_{j=1}^{N-1} d_j$$

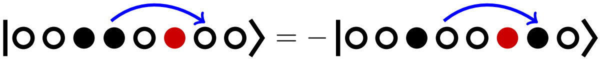
Fermionic operators

$c_i^\dagger c_j$, which acts as follows

$$c_i^\dagger c_j \left|\psi\right\rangle = \epsilon\, \sigma_i^+ \sigma_j^- \left|\psi\right\rangle$$

$$\epsilon = (-1)^{\text{Number of particles jumped over}}$$

For example

$c_7^\dagger c_4$ 

Obviously, $c_i^\dagger c_{i+1} = \sigma_i^+ \sigma_{i+1}^-$, but only true for nearest neighbors.

Well known quantum Hamiltonians

- The anisotropic Heisenberg spin chain (or XXZ spin chain)

$$H = -\Delta D + \sum_{j=1}^{N-1} \left(\sigma_j^- \sigma_{j+1}^+ + \sigma_j^+ \sigma_{j+1}^- \right)$$

  - Quantum integrable, related to the six vertex model.
  - The term proportional to D is sometimes called *interactions*.
  - In case $\Delta = 1$, H coincides with the generator for SSEP.

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- Free fermions with next nearest neighbor hoppings

$$H = \sum_j \left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right) + \nu \left(c_j^\dagger c_{j+2} + c_{j+2}^\dagger c_j \right)$$

where $\nu \geq 0$ in this talk.

Infinite Hamiltonian with domain wall boundary conditions

Reference state $|\psi\rangle$, with all sites filled for $j \leq 0$, empty for $j > 0$.

$$|\psi\rangle = \Big| \cdots \bullet \bullet \bullet \bullet \circ \circ \circ \circ \cdots \Big\rangle$$

$|\psi_{x_l,\ldots,x_1}\rangle$ obtained from $|\psi\rangle$ by moving particles at positions $-l+1,\ldots,0$ to positions $-l+1 \leq x_l < \ldots < x_1$. For example

$$|\psi_{02}\rangle = \Big| \cdots \bullet \bullet \circ \bullet \circ \bullet \circ \circ \cdots \Big\rangle$$

Can make sense of

$$H = -\Delta D + \sum_{j \in \mathbb{Z}} \left(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ \right)$$

over the space of states spanned by the $|\psi_{x_l,\ldots,x_1}\rangle$ for $l \geq 0$. For example

$$\begin{aligned} H\,|\psi\rangle &= H \,\big| \cdots \bullet \bullet \bullet \bullet \circ \circ \circ \circ \cdots \big\rangle \\ &= -\Delta\,|\psi\rangle + |\psi_1\rangle \end{aligned}$$

$$H^2\,|\psi\rangle \quad = (1+\Delta^2)\,|\psi\rangle - 4\Delta\,|\psi_1\rangle + |\psi_2\rangle + |\psi_{01}\rangle$$

and more generally, objects such as $e^{\tau H}|\psi\rangle$ for any $\tau \in \mathbb{C}$.

Transfer matrix for the six vertex model

[Lieb, Baxter,...]

$$T_{\mathrm{e}} = \prod_{j \text{ even}} \left[a + b(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+) + (c-a) d_j \right]$$

$$T_{\mathrm{o}} = \prod_{j \text{ odd}} \left[a + b(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+) + (c-a) d_j \right]$$

The partition function in the pictures at the beginning is e.g.

$$Z = \langle \psi | (T_{\mathrm{e}} T_{\mathrm{o}})^n | \psi \rangle$$

Remember $a = 1$, $\Delta = \cos\gamma$ fixed.

$$T(b) = T_{\mathrm{e}} T_{\mathrm{o}}$$

Can show using the Lie-Trotter formula

$$\lim_{n\to\infty} \left[T\left(\frac{\tau \sin\gamma}{n} \right) \right]^n = e^{\tau H}$$

where H is the Hamiltonian of the XXZ spin chain.

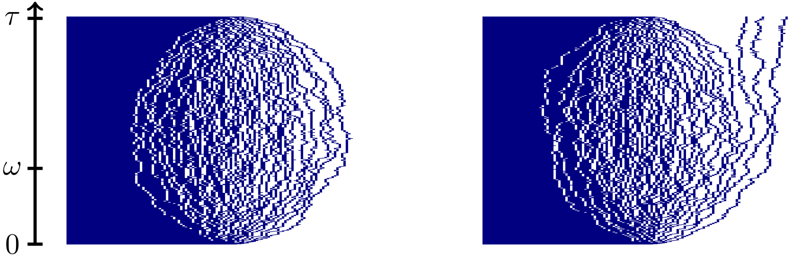
Can borrow results from the six vertex model with domain wall boundary conditions, obtained using integrability techniques.
[Korepin, Zinn-Justin, Colomo, Pronko, Sportiello,. . .]

Main results

- $\mathcal{A}(\tau) = \langle \psi | e^{\tau H} | \psi \rangle$ as a Fredholm determinant. [JMS 2017]
- Formulas for all $\mathcal{A}_{x_l,\ldots,x_1}(\tau) = \langle \psi_{x_l,\ldots,x_1} | e^{\tau H} | \psi \rangle$. [JMS 2021]

Probabilities of particle occupancies in the Hamiltonian limit of the six vertex model. For some $\tau > 0$ and $\omega \in [0, \tau]$

$$P_{\omega,\tau}(x_l, \ldots, x_1) = \frac{\mathcal{A}_{x_l,\ldots,x_1}(\omega)\mathcal{A}_{x_l,\ldots,x_1}(\tau - \omega)}{\mathcal{A}(\tau)}$$



For $\tau \geq 0$ all amplitudes are positive, so this defines a legitimate probabilistic model. Limit shapes in the scaling limit.

Another very interesting problem (for $t \geq 0$):

$$P_t(x_l, \ldots, x_1) = |\mathcal{A}_{x_l,\ldots,x_1}(it)|^2$$

Real-time evolution of the quantum system, with initial state $|\psi\rangle$. Conjectures from generalized hydrodynamics in the scaling limit. [Castro-Alvaredo, Doyon, Yoshimura 2016] [Bertini, Collura, De Nardis, Fagotti 2016].

Simple analytical solution for the density profile studied here ($|\Delta| < 1$) [Collura, De Luca, Viti 2017]

$$\mathcal{A}\left(\frac{\tau}{\sin\gamma}\right) = e^{-\frac{\tau^2}{6}} \exp\left(\sum_{n\geq 1} \frac{1}{n} \int_{\mathbb{R}^n} V(x_1,x_2)\dots V(x_n,x_1) dx_1 \dots dx_n\right)$$

$$V(x,y) = \frac{\sqrt{\tau y} J_0(2\sqrt{\tau x}) J_0'(2\sqrt{\tau y}) - \sqrt{\tau x} J_0(2\sqrt{\tau y}) J_0'(2\sqrt{\tau x})}{2(x-y)} [\Theta(y) - w_0(y)]$$

with

$$w_0(y) = \frac{1-e^{-\gamma y}}{1-e^{-\pi y}}$$

follows from a result of [Slavnov 2003].

$h(\tau|z) = \dfrac{1}{\mathcal{A}(\tau)} \sum_{x \geq 0} \mathcal{A}_x(\tau) z^x$ satisfies the exact PDE [JMS 2021]

$$\left(\tau\partial_\tau^2 + \left[1 - 2\tau\left(\frac{1}{z} - \Delta\right)\right]\partial_\tau + Q(\tau) - z + \Delta\right) h = \left(1 - 2\Delta z + z^2\right)\partial_z h$$

where $Q(\tau) = 2\tau \frac{d^2 \log \mathcal{A}(\tau)}{d\tau^2} + \frac{d \log \mathcal{A}(\tau)}{d\tau}$.

All amplitudes are given by

$$\frac{\mathcal{A}_{x_l,\dots,x_1}(\tau)}{\mathcal{A}(\tau)} = \oint_{\mathcal{C}^l} \prod_{j=1}^{l} \frac{dz_j}{2\mathrm{i}\pi z_j^{x_j+l}} \frac{\det\limits_{1\leq j,k\leq l}\left(z_k^{l-j}\left[1 - z_k\partial_\tau\right]^{j-1} h(\tau|z_k)\right)}{\prod\limits_{1\leq j<k\leq l}\left(z_j z_k - 2\Delta z_k + 1\right)}$$

Free/determinantal case $\Delta = 0$

Partition function $Z(\tau) = \mathcal{A}(\tau) = e^{\tau^2/2}$.

Solution to the PDE: $h(\tau|z) = e^{\tau z}$.

Yields after some manipulations

$$\frac{\mathcal{A}_{x_l,\ldots,x_1}(\tau)}{\mathcal{A}(\tau)} = \det_{1\leq j,k\leq l}\left(\oint_{\mathcal{C}} \frac{dz\, e^{\tau z}}{2\mathrm{i}\pi z^{x_j+k}}\right)$$

Relation to PNG droplet model [Praehoffer Spohn 2001], Poissonized Plancherel measures [Baik, Deift, Johansson 1998], Gross-Witten-Wadia matrix model, ...

Case $|\Delta| < 1$

Large τ asymptotics.

$$h(\tau|z) = e^{\tau F(z)} O(1)$$

where F can be found explicitly. This allows to compute the "energy" function

$$\frac{1}{\tau} \log \mathcal{A}_{X\tau}(\tau) \to G(X)$$

Can reconstruct the full arctic curve using the tangent method [Colomo, Sportiello 2016]

Real time conjectures

$\mathcal{R}(t) = |\mathcal{A}(it)|^2$ gives the exact return probability. [JMS 2017]

- $\Delta = \cos\gamma$, $\gamma = \frac{\pi p}{q}$

$$-\log \mathcal{R}(t) = \left(\frac{q^2}{(q-1)^2} - 1\right)\frac{(t\sin\gamma)^2}{12} + o(t)$$

- $\Delta = \cos\gamma$, $\frac{\gamma}{\pi} \notin \mathbb{Q}$

$$-\log \mathcal{R}(t) = (\sin\gamma)t + o(t)$$

- $|\Delta| = 1$

$$-\log \mathcal{R}(t) = \zeta(3/2)\sqrt{t/\pi} - \frac{1}{2}\log t + o(\log t)$$

Log-enhanced diffusion. [Ljubotina, Znidaric, Prosen 2017] [Misguich, Mallick, Krapivsky 2017], [Gamayun, Miao, Ilievski 2019], [Misguich, Pavloff, Pasquier 2019].

[Bocini, JMS 2021]

Makes sense to look at other quantum models in imaginary time, including fermionic chains.

$$H = \frac{1}{2}\sum_j \left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j\right) + \nu\left(c_j^\dagger c_{j+2} + c_{j+2}^\dagger c_j\right)$$

dispersion $\varepsilon(k) = \cos k + \nu\cos 2k$ ($\nu \geq 0$ here).

Total positivity for free fermions Hamiltonians

$\langle\phi|e^{\tau H}|\phi'\rangle \geq 0$ for any local particle states $|\phi\rangle, |\phi'\rangle$ iff

$\varepsilon(k)$ is a linear combination of

$$1 \quad , \quad e^{\mathrm{i}k} \quad , \quad e^{-\mathrm{i}k}$$

with positive coeffcients, and the

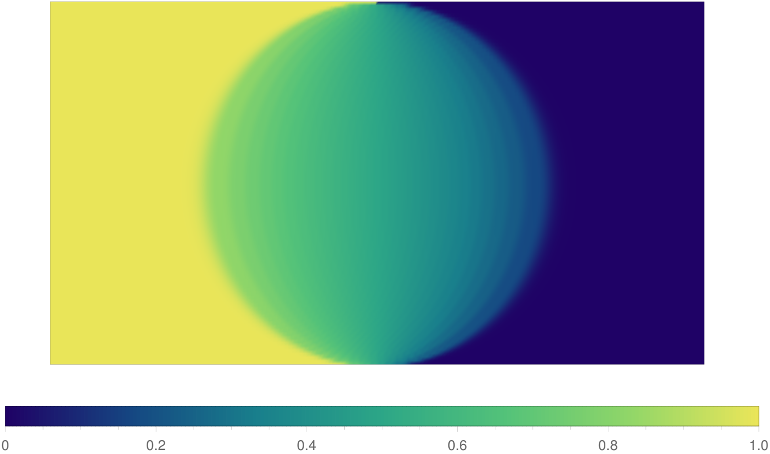
$$\log(1+\alpha e^{ik}) \quad , \quad \log(1+\beta e^{-ik}) \quad , \quad \log\frac{1}{1-\gamma e^{ik}} \quad , \quad \log\frac{1}{1-\delta e^{-ik}}$$

with positive integer coefficients, and $\alpha, \beta, \gamma, \delta \geq 0$.

[Edrei 1952, Thoma 1964]

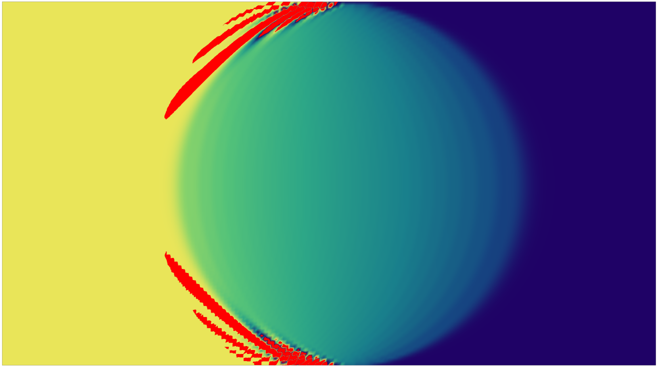
Average density profile for $\varepsilon(k) = \cos k$

Previously studied in relation to growth models [Prähofer, Spohn 2000]

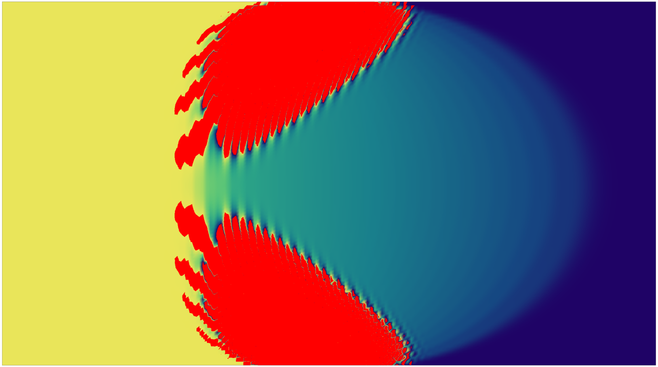


Density is frozen (to 1 or 0) outside an “arctic” circle.

$\nu = \frac{1}{15}$, new “crazy regions” in red with density not in $[0, 1]$.



$\nu = \frac{1}{4}$, new "crazy regions" in red with density not in $[0, 1]$.



Sign issues

$$e^{\tau H}|\psi\rangle = \sum_{\mathcal{C}} a_{\mathcal{C}}(\tau)|\mathcal{C}\rangle \qquad , \qquad a_{\mathcal{C}}(\tau) = \langle \mathcal{C}|e^{\tau H}|\psi\rangle$$

$$a_{\mathcal{C}}(\tau) = \sum_{m=0}^{\infty} \frac{\tau^m}{m!} \langle \mathcal{C}|H^m|\psi\rangle \,.$$

$$\begin{aligned} H|\psi\rangle &= H|..1111100000..\rangle \\ &= |..1111010000..\rangle + \nu|..1111001000..\rangle - \nu|..1110110000..\rangle \end{aligned}$$

so for sufficiently small τ, some $a_{\mathcal{C}}(\tau)$ are negative. Hence

$$\mathbb{P}(\mathcal{C},\omega,\tau) = \frac{a_{\mathcal{C}}(\omega)a_{\mathcal{C}}(\tau-\omega)}{\sum_{\mathcal{C}} a_{\mathcal{C}}(\omega)a_{\mathcal{C}}(\tau-\omega)}$$

can be negative (if $\omega \neq \tau/2$).

More general wall states

$|\psi^{1/n}\rangle$ for $n \in \{1, 2, 3 \ldots\}$

$$|\psi^{1}\rangle = |\ldots 111111111000000 \ldots\rangle$$

$$|\psi^{1/2}\rangle = |\ldots 101010101000000 \ldots\rangle$$

$$|\psi^{1/3}\rangle = |\ldots 100100100100000 \ldots\rangle$$

One fermion every n-th site, then no fermions.

A new exact formula ($y \in [-R, R]$ instead of $\omega \in [0, \tau]$)

$$n_x(y) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi+\mathrm{i}\eta}^{\pi+\mathrm{i}\eta} \frac{dq}{2\pi} \frac{e^{\Phi_n(k,x,y)-\Phi_n(q,x,y)} e^{\Omega_n(k)+\Omega_n(q)}}{1-e^{-\mathrm{i}n(k-q)}}$$

$$\Phi_n(k,x,y) = -\mathrm{i}kx - y\varepsilon(k) + \mathrm{i}R\tilde{\varepsilon}_n(nk),$$

$$\Omega_n(k) = R\left[\varepsilon(k) - \varepsilon_n(nk)\right]$$

$$\varepsilon_n(k) = \frac{1}{2R} \log\left(\frac{1}{n} \sum_{p=0}^{n-1} e^{2R\varepsilon\left(\frac{k+2p\pi}{n}\right)} \right)$$

$\tilde{\varepsilon}_n$ denotes the periodic Hilbert transform of ε_n.

This formula works only for the initial states $|\psi^{1/n}\rangle$. Saddle point analysis gives a normal density profile.

Dilution argument

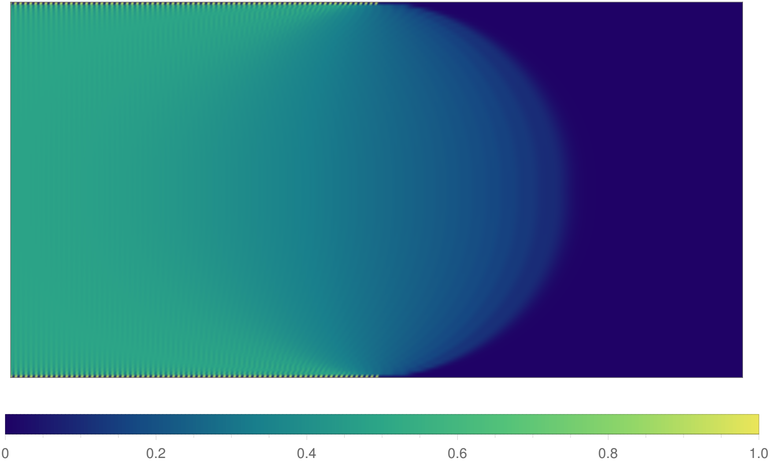
Minus signs occur when one fermion hop around another, e.g.

$$|1010\overset{\curvearrowright}{110}111\rangle$$

so if one thinks of density as reasonably smooth, minus signs are only generated in regions with high –but not too high– densities.

Makes sense to look at lower density boundary conditions, such as $|\psi^{1/2}\rangle, |\psi^{1/3}\rangle$, etc.

Density profile (exact simulations for $|\psi^{1/2}\rangle$)



No sign of crazy region, for finite τ, any $\omega \in [0, \tau]$.

Conclusions (fermionic chain)

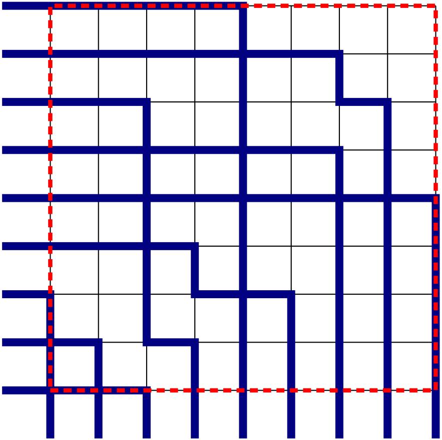
- Always positive for $y = 0$. Edge behavior is interesting [Betea, Bouttier, Walsh 2020] related to higher order Tracy-Widom behavior [Di Francesco, Ginsparg, Zinn-Justin 1995] [Akemann, Atkin 2012] [Le Doussal, Majumdar, Schehr 2018].
- There are many (weaker) forms of positivity.
- Similar story in the presence of several bands.
- Presumably similar story in the presence of interactions (add higher order charges to the XXZ Hamiltonian).

Conclusions (XXZ chain)

- More can be extracted from those formulas in imaginary time.
- Real time asymptotics and connection to hydrodynamics?
- Exact formula for the emptiness formation probability, gives access to the distribution of the rightmost particle. KPZ or not KPZ.
- Alternative methods to compute the amplitudes from coordinate Bethe Ansatz [Saenz, Tracy, Widom 2022] or the F basis [Feher, Pozsgay 2019].

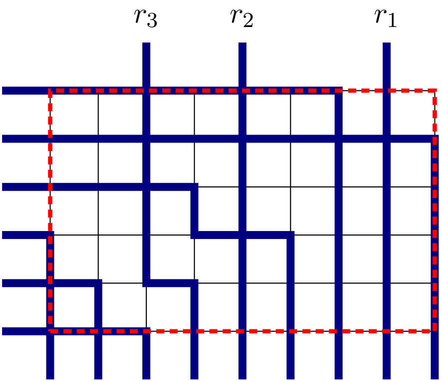
Thank you!

Some steps in the derivation.



Partition function: Izergin-Korepin $n \times n$ determinant.

Hamiltonian limit: Fredholm determinant.



[Colomo, Pronko 2007] [Cantini, Colomo, Pronko 2019] [Colomo, Di Giulio, Pronko 2021].

Requires the knowledge of polynomials $p(n, \epsilon|x)$ which are orthonormal wrt the weights

$$w_\epsilon(x) = e^{-\epsilon x} w_0(x) \qquad , \qquad w_0(x) = \frac{1 - e^{-\gamma x}}{1 - e^{-\pi x}}$$

Orthogonal polynomials

The limit

$$q(\alpha|x) = \lim_{n\to\infty} \sqrt{\frac{n}{\alpha}}\, p(n, \alpha/n|x)$$

satisfies the ODE

$$\left[\alpha\partial_\alpha^2 + \partial_\alpha + f(\alpha) + x\right] q(\alpha|x) = 0$$

where

$$f(\alpha) = 2\alpha \frac{d^2 \log \mathcal{Y}(\alpha)}{d\alpha^2} + \frac{d \log \mathcal{Y}(\alpha)}{d\alpha}$$

is given in terms of the Fredholm determinant

$$\mathcal{Y}(\alpha) = \det(I - V)_{L^2(\mathbb{R})}$$

$$V(x,y) = \frac{\sqrt{y} J_0(2\sqrt{x}) J_0'(2\sqrt{y}) - \sqrt{x} J_0(2\sqrt{y}) J_0'(2\sqrt{x})}{(x-y)} \left[\Theta(y) - w_0\left(\frac{y}{\alpha}\right)\right]$$