The dSKP recurrence: Combinatorial aspects and geometric systems

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joint works with

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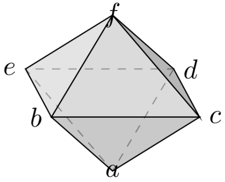
Outline

- The dSKP equation + recurrence
- Combinatorial solution I
- Geometric applications
- Devron property + combinatorial solution II
- Geometric applications

The discrete Schwarzian Kadomtsev-Petviashvili equation (dSKP)



$$\frac{a-b}{b-c} \times \frac{c-f}{f-d} \times \frac{d-e}{e-a} = -1.$$

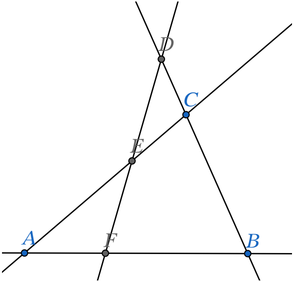
- Discrete version of Schwarzian KP equation, related to soliton theory [Bogdanov, Konopelchenko '98],
- Discrete version of quasi-conformal maps [Konopelchenko, Schief '01],
- Many geometric systems.

Menelaus' theorem

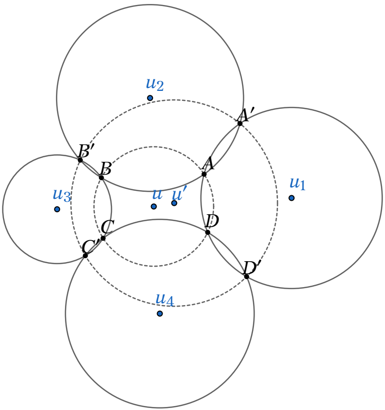
Theorem (Menelaus (~100 AD))

Let ABC be a triangle, and let D be a point on (BC), E on (AC) and F on (AB).
Then D, E, F are on a line iff, in complex coordinates,

$$\frac{a-f}{f-b} \times \frac{b-d}{d-c} \times \frac{c-e}{e-a} = -1.$$



Miquel's theorem



Theorem (Miquel '1838)

Given the four external circles, the points A, B, C, D are concyclic iff the points A', B', C', D' are.

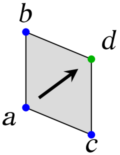
Theorem (Affolter '21, Kenyon-Lam-Ramassamy-Russkikh '22)

If the above holds, the circle centers satisfy

$$\frac{u - u_1}{u_1 - u_2} \times \frac{u_2 - u'}{u' - u_3} \times \frac{u_3 - u_4}{u_4 - u} = -1.$$

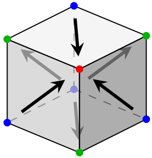
⇝ Related to *t*-realizations of the dimer model [KLRR, Chelkak-Laslier-Russkikh '20], Miquel's dynamics on the dimer model [Ramassamy '20; KLRR].

CLASSIFICATION OF CONSISTENT EQU. *[Adler-Bobenko-Suris '03, '11]*



$$F(a, b, c, d) = 0$$

An equation on quads, $F(a, b, c, d) = 0$, is said to be consistent if it propagates uniquely on a cube:

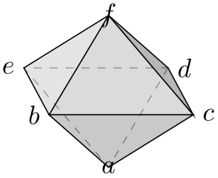


⇝ Analogous definition for equations on cubes, and on octahedra.

Classification of consistent equations

Theorem (Adler-Bobenko-Suris '11)

Up to admissible transformations, the consistent equations on octahedra are:



$$\chi_1 : f = \frac{bd + ce}{a}$$

$$\chi_2 : f = \frac{cbd - bda - cbe - cde + bde + cae}{bd + ca - ba - da - ce + ae}$$

$$\chi_3 : f = \frac{cb - ca + da - de}{b - e}$$

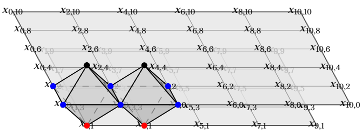
$$\chi_4 : f = \frac{bda + cbe - bde - cae}{a(b - e)}$$

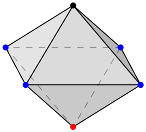
$$\chi_5 : f = \frac{bda + cbe - cae}{ba}$$

- χ_1 is dKP.
- χ_2 is dSKP.
- χ_3, χ_4, χ_5 can be obtained from χ_2 by taking appropriate limits

The dSKP recurrence

- Octahedral-tetrahedral lattice: $\mathscr{L} = \{(i,j,k) \in \mathbb{Z}^3 : i+j+k \in 2\mathbb{Z}\}$.





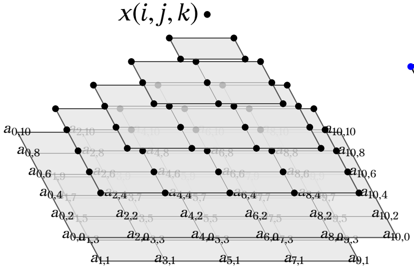
- A function $x : \mathscr{L} \to \hat{\mathbb{C}}$ satisfies the dSKP recurrence if

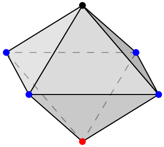
$$\frac{(x_{-e_3} - x_{e_2})(x_{-e_1} - x_{e_3})(x_{-e_2} - x_{e_1})}{(x_{e_2} - x_{-e_1})(x_{e_3} - x_{-e_2})(x_{e_1} - x_{-e_3})} = -1,$$

where $x_q(p) := x(p+q)$ for every $q \in \{e_i\}_{i=1}^3$, and the relation is evaluated at any $p \in \mathbb{Z}^3 \setminus \mathscr{L}$.

The dSKP recurrence

- Consider initial conditions $a = (a_{i,j})_{(i,j)\in\mathbb{Z}^2}$ at levels 0 and 1, *i.e.*, at vertices $(i, j, [i + j]_2)_{(i,j)\in\mathbb{Z}^2}$ of $\mathscr{L}$.





- What is the value of $x(i, j, k)$, $k \geq 1$, in terms of the initial conditions?

dSKP - Solution I

Theorem (Affolter-dT-Melotti '22)

Let $x : \mathscr{L} \to \hat{\mathbb{C}}$ be a function satisfying the dSKP recurrence with initial conditions $a = (a_{i,j})_{(i,j)\in\mathbb{Z}^2}$ at vertices $(i,j,[i+j]_2)_{(i,j)\in\mathbb{Z}^2}$. Then, for all points $(i,j,k) \in \mathscr{L}$ such that $k \geq 1$,

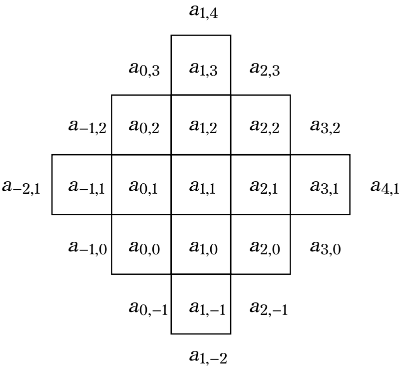
$$x(i,j,k) = Y(A_{k-1}[a_{i,j}], a),$$

where

$$Y(A_{k-1}[a_{i,j}], a) := C(A_{k-1}[a_{i,j}]) \frac{Z_{\overrightarrow{\dim}}(A_{k-1}[a_{i,j}], a^{-1}, \varphi)}{Z_{\overrightarrow{\dim}}(A_{k-1}[a_{i,j}], a, \varphi)}.$$

dSKP - Solution I: Aztec diamond $A_k[a_{i,j}]$

- $A_{k-1}[a_{i,j}]$: Aztec diamond of size $k-1$ centered at $a_{i,j}$.

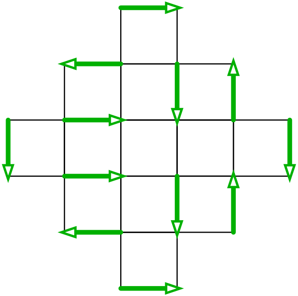


Example: Aztec diamond $A_3[a_{1,1}]$ used to compute $x(1,1,4)$.

$\rightsquigarrow$ Aztec diamond with face weights.

dSKP - Solution I: partition function $Z_{\overrightarrow{\text{dim}}}(A, a, \varphi)$

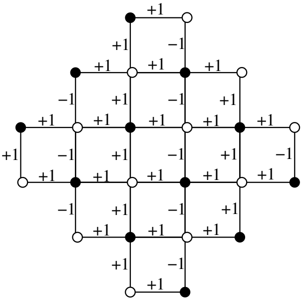
- Oriented dimer configuration $\vec{\mathsf{M}}$ of Aztec diamond A: unoriented version M is a dimer configuration, *i.e.*, subset of edges s.t. every vertex is incident to exactly one edge of $\mathsf{M} \rightsquigarrow \vec{\mathscr{M}}$.



dSKP - Solution I: partition function $Z_{\overrightarrow{\mathrm{dim}}}(A, a, \varphi)$

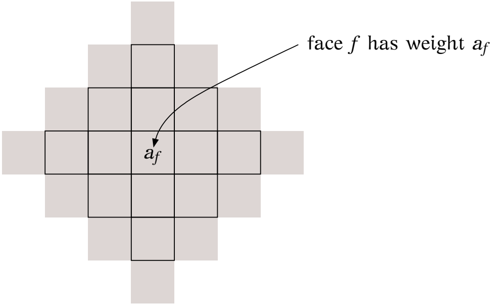
- Kasteleyn orientation $\varphi \in \{-1, 1\}^{\vec{E}}$: skew-symmetric function on $\vec{E}$ such that, for every inner face f,

$$\prod_{(w,b)\in\partial f} \varphi_{(w,b)} = -1.$$



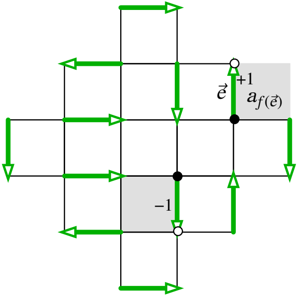
dSKP - Solution I: partition function $Z_{\overrightarrow{\dim}}(A, a, \varphi)$

- Aztec diamond A with face weights $(a_f)_{f \in F}$.



dSKP - Solution I: partition function $Z_{\overrightarrow{\dim}}(A, a, \varphi)$

- Every oriented dimer edge has
  - a Kasteleyn weight: $\varphi_{\vec{e}}$
  - a face weight on the right: $a_{f(\vec{e})}$



- The partition function $Z_{\overrightarrow{\dim}}(A, a, \varphi)$:

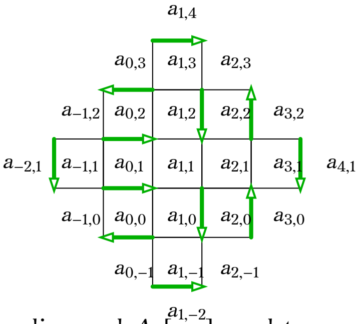
$$Z_{\overrightarrow{\dim}}(A, a, \varphi) = \sum_{\vec{M} \in \vec{\mathscr{M}}} \prod_{\vec{e} \in \vec{M}} \varphi_{\vec{e}} \; a_{f(\vec{e})}.$$

dSKP - Solution I

Theorem (Affolter-dT-Melotti '22)

Let $x : \mathscr{L} \to \hat{\mathbb{C}}$ be a function satisfying the dSKP recurrence with initial conditions $a = (a_{i,j})_{(i,j)\in\mathbb{Z}^2}$ at vertices $(i,j,[i+j]_2)_{(i,j)\in\mathbb{Z}^2}$. Then, for all points $(i,j,k) \in \mathscr{L}$ such that $k \geq 1$,

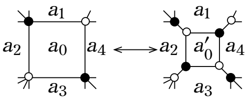
$$x(i,j,k) = C(A_{k-1}[a_{i,j}]) \frac{Z_{\overrightarrow{\dim}}(A_{k-1}[a_{i,j}], a^{-1}, \varphi)}{Z_{\overrightarrow{\dim}}(A_{k-1}[a_{i,j}], a, \varphi)} =: Y(A_{k-1}[a_{i,j}], a).$$



Example: Aztec diamond $A_3[a_{1,1}]$ used to compute $x(1,1,4)$.

dSKP - Solution I: idea of proof [Speyer '07]

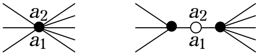
- Show that for any bipartite planar G with face weights $(a_f)_{f \in F}$, the ratio of partition functions $Y(\mathsf{G}, a)$ is invariant when:
  - an urban renewal is performed and weights satisfy dSKP



and

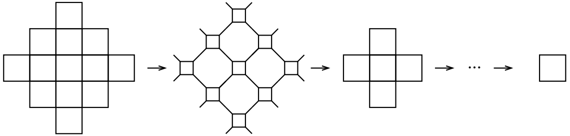
$$\frac{a_0 - a_1}{a_1 - a_2}\,\frac{a_2 - a_0'}{a_0' - a_3}\,\frac{a_3 - a_4}{a_4 - a_0} = -1$$

  - a contraction/expansion of a degree 2 vertex



- Then, do an induction

$Y(A_{k-1}[a_{i,j}], a)$



$x(i, j, k)$

Previous results of the same kind

- dKP/Octahedron recurrence/Dodgson condensation: [Speyer '07] $\rightsquigarrow$ dimer model
- Cube recurrence: [Caroll, Speyer '04] $\rightsquigarrow$ cube groves
- Hexahedron recurrence: [Kenyon, Pemantle '16] $\rightsquigarrow$ double dimer model
- Kashaev's recurrence: [Melotti '18] $\rightsquigarrow$ C_2^1−loop model

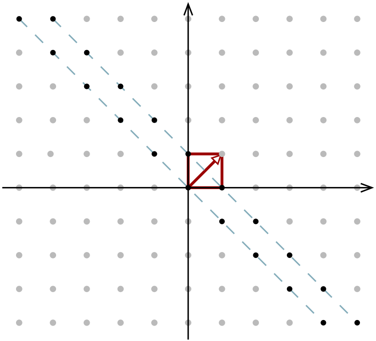
All of the above solutions are Laurent polynomials in the initial data, with coefficients equal to 1 in the first 2 cases [Fomin, Zelevinsky '02].

Application to discrete holomorphic functions

A function $z : \mathbb{Z}^2 \to \hat{\mathbb{C}}$ is discrete holomorphic [Bobenko-Pinkall '96] if, for every quad of $\mathbb{Z}^2$,

$$\frac{z_{i,j} - z_{i+1,j}}{z_{i+1,j} - z_{i+1,j+1}} \frac{z_{i+1,j+1} - z_{i,j+1}}{z_{i,j+1} - z_{i,j}} = -1.$$

The data of $(z_{i,j})_{i+j \in \{0,1\}}$ is enough to determine all of z.



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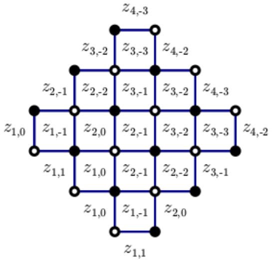
Theorem (A-dT-M '22)

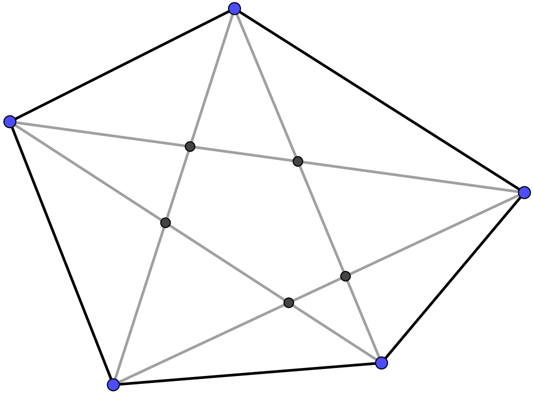
For every $(i,j) \in \mathbb{Z}^2$ such that $i + j \geq 2$, there is an Aztec diamond A_{i+j-2} with explicit weights $a = (z_{k,\ell})_{k+\ell \in \{0,1,2\}}$ such that

$$z_{i,j} = Y(A_{i+j-2}, a).$$



Aztec diamond used for $z_{4,1}$.

Application to pentagram map [Schwartz '92]



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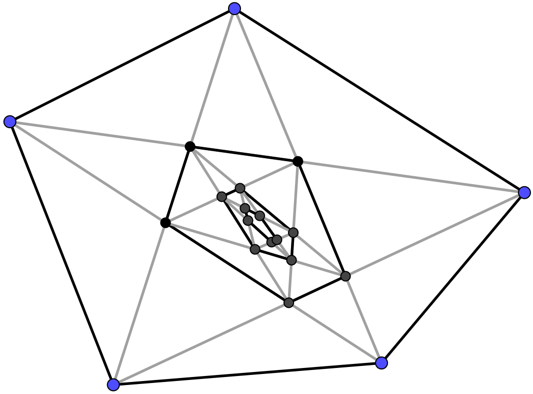

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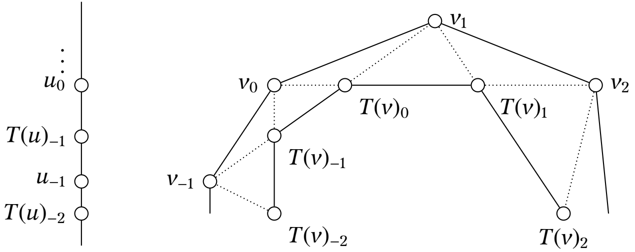
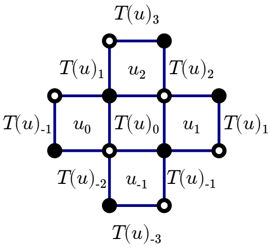


Theorem (A-dT-M '22)

For all $j \geq 1$ and $i \in \mathbb{Z}$, there is an Aztec diamond A_{j-1} with explicit weights $a = (u_k, T(u)_\ell)_{k,\ell \in \mathbb{Z}}$ such that

$$T^j(u)_i = Y(A_{j-1}, a).$$

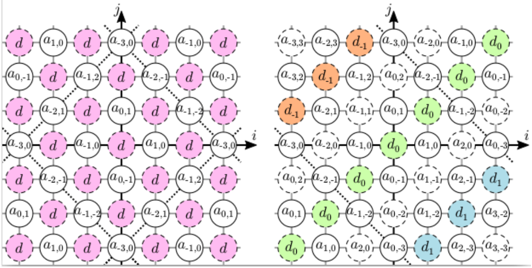


Example for $T^3(u)_{-1}$.

Remark: [Glick '11] has an explicit expression using F-polynomials of cluster algebras.

Devron property

- Devron property [Glick '15]: for a birational integrable dynamics T, if some data is singular for T^{-1}, then there should be an $n \in \mathbb{N}$ such that it is singular for T^n.
- Singular data for backwards dSKP:
  1. (m, m)-periodic + every p-th diagonal at height 0 is constant.
  2. $(m, m) + (m, -m)$-periodic + diagonals at height 0 are constant + rows at height 1 are translates of each other.



Devron property

1. (m,m)-periodic + every p-th diagonal at height 0 is constant.
2. $(m,m)+(m,-m)$-periodic + diagonals at height 0 are constant + rows at height 1 are translates of each other.

Theorem (Affolter-dT-Melotti '22)

Under assumptions 1, after $mp-2p+1$ iterations of the dSKP recurrence, again every p diagonal is constant.

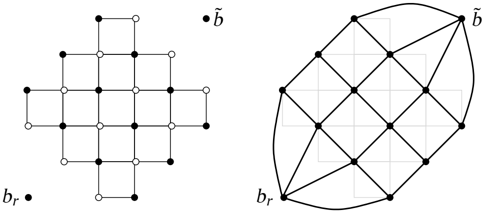
Theorem (Affolter-dT-Melotti '22)

Under assumptions 2, after $m-1$ iterations of the dSKP recurrence, the value is constant and equal to the shifted harmonic mean of the initial data:

$$\forall (i,j,m) \in \mathscr{L}, \quad x(i,j,m) = d + \left(\frac{1}{m} \sum_{i=0}^{m-1} \frac{1}{a_{i,1-i} - d} \right).$$

dSKP solution II: complementary trees/forests

- Consider Aztec diamond A_k with two additional black vertices $\{b_r, \tilde{b}\}$.
- From A_k construct the graph $A_k^\bullet$ as follows.



A complementary tree/forest of $A_k^\bullet$ is a pair (T, F) of edge config. s.t.

- T is a spanning tree of $A_k^\bullet$ rooted at b_r,
- F is a spanning forest of $A_k^\bullet$ rooted at $b_r, \tilde{b}$,
- $\mathsf{T} \cap \mathsf{F} = \emptyset$.

$\rightsquigarrow \mathscr{F}$: set of complementary trees/forests of $A_k^\bullet$.

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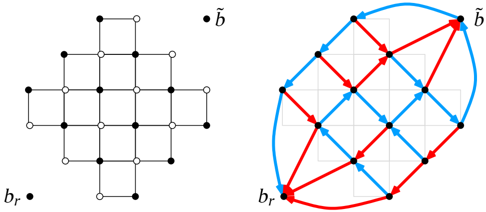

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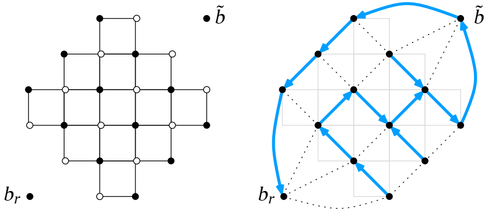

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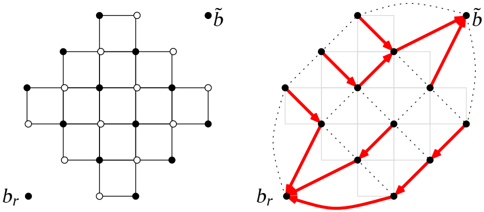

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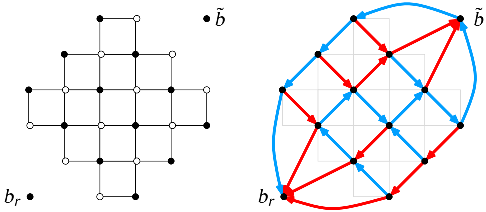

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dSKP solution II: complementary trees/forests

Theorem (Affolter-dT-Melotti '22)

Consider Aztec diamond A_k with face weights $a = (a_f)_{f \in F}$. For any Kasteleyn orientation φ,

$$Z_{\overrightarrow{\mathrm{dim}}}(A_k, a, \varphi) = \pm \sum_{(\mathsf{T},\mathsf{F}) \in \mathscr{F}} \mathrm{sgn}(\mathsf{T}, \mathsf{F}) \prod_{\vec{e} \in \mathsf{F}} a_{f_{\vec{e}}}.$$

Moreover, there is a bijection between terms in the sum in the r.h.s. and monomials of $Z_{\overrightarrow{\mathrm{dim}}}(A_k, a, \varphi)$ in the a variables.

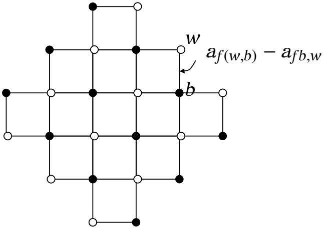
Remark

Result in the spirit of [Speyer] [Caroll, Speyer] [Kenyon, Pemantle] [Melotti] giving a combinatorial interpretation of monomials.

dSKP solution II: idea of proof

- Weighted adjacency matrix $K(a)$ of A_k:

$$\forall\, w \in W,\ b \in B, \quad K(a)_{w,b} = \begin{cases} a_{f(w,b)} - a_{f(b,w)} & \text{if } w \sim b \\ 0 & \text{otherwise.} \end{cases}$$



Proposition (A-dT-M)

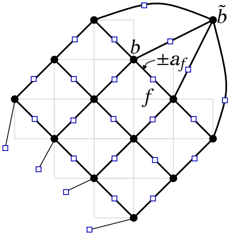
For any Kasteleyn orientation φ of A_k, there exists $\epsilon(\varphi) \in \{-1, 1\}$ s.t.

$$Z_{\overrightarrow{\dim}}(A_k, a, \varphi) = \epsilon(\varphi)\, \det(K(a)).$$

dSKP solution II: idea of proof

- Consider the matrix $C(a)$,

$$\forall f \in F,\ b \in B \cup \{\tilde{b}\}, \quad C(a)_{f,b} = \begin{cases} \pm a_f & \text{if } b \text{ belongs to } \partial f, \\ 0 & \text{otherwise.} \end{cases}$$



Proposition (A-dT-M)

$$\det(K(a)) = \pm \det\begin{pmatrix} C(1)^{\tilde{b}} & C(1)^{B} & C(a)^{B} \end{pmatrix}.$$

dSKP solution II: idea of proof + consequences

- The relation

$$Z_{\overrightarrow{\dim}}(A_k, a, \varphi) = \pm \det \begin{pmatrix} C(1)^{\tilde{b}} & C(1)^B & C(a)^B \end{pmatrix} \qquad (1)$$

  + extra work allows to prove compl. trees/forests representation.
- Relation (1) + extra work allows to prove the following.

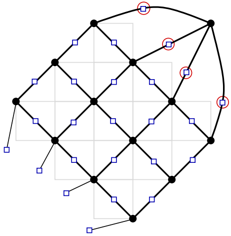
Theorem (A-dT-M '22)

Let $v \in \mathbb{C}^F$ be a non-zero vector such that

$$\begin{pmatrix} C(1)^B & C(a)^B \end{pmatrix}^T v = 0.$$

Let $(v_i)_{i=0}^k$ be the elements corresponding to ∘. Then,

$$Y(A_k, a) = \frac{\sum_{i=0}^k a_{f_i} v_i}{\sum_{i=0}^k v_i}.$$



Both theorems + extra work ⇒ Devron properties.

Application to P-nets

A map $p : \mathbb{Z}^2 \to \hat{\mathbb{C}}$ is a P-net [Bobenko-Pinkall '99] if,

$$\forall (i,j) \in \mathbb{Z}^2, \quad \frac{1}{p_{i+1,j} - p_{i,j}} - \frac{1}{p_{i,j+1} - p_{i,j}} + \frac{1}{p_{i-1,j} - p_{i,j}} - \frac{1}{p_{i,j-1} - p_{i,j}} = 0.$$

The data of $(p_{i,j})_{j\in\{0,1\}}$ determines p.

Theorem (Glick '15, Yao '14)

Let $m \geq 1$, and let p be an m-periodic P-net such that, for all $i \in \mathbb{Z}$, $p_{i,0} = 0$. Then, if the following are well defined,

$$\forall i \in \mathbb{Z}, \quad p_{i,m} = \left(\frac{1}{m} \sum_{\ell=0}^{m-1} p_{\ell,1}^{-1} \right)^{-1},$$

that is the singularity repeats after $m-1$ steps, and its value is the harmonic mean of $(p_{i,1})$.

$\rightsquigarrow$ We provide an alternative proof.

Application to P-nets

A map $p : \mathbb{Z}^2 \to \hat{\mathbb{C}}$ is a P-net [Bobenko-Pinkall '99] if,

$$\forall\,(i,j) \in \mathbb{Z}^2, \qquad \frac{1}{p_{i+1,j} - p_{i,j}} - \frac{1}{p_{i,j+1} - p_{i,j}} + \frac{1}{p_{i-1,j} - p_{i,j}} - \frac{1}{p_{i,j-1} - p_{i,j}} = 0.$$

The data of $(p_{i,j})_{j \in \{0,1\}}$ determines p.

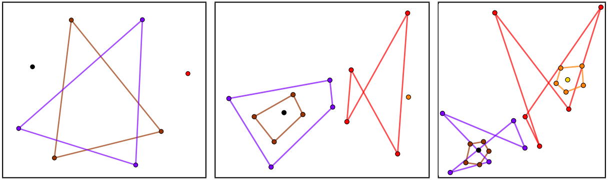


Illustration. The black dot is at 0 and corresponds to $p_{i,0}$; the brown dots are the values of $p_{i,1}$. Those are m-periodic with $m = 3$ (left), resp. $m = 4$ (center), resp. $m = 5$ (right).

Application to integrable cross-ratio maps

Let $\alpha, \beta : \mathbb{Z} \to \mathbb{C} \setminus \{0\}$. An integrable cross-ratio map [Bobenko-Pinkall '99] is a map $z : \mathbb{Z}^2 \to \hat{\mathbb{C}}$ such that,

$$\forall\, (i,j) \in \mathbb{Z}^2, \qquad \frac{(z_{i,j} - z_{i+1,j})(z_{i+1,j+1} - z_{i,j+1})}{(z_{i+1,j} - z_{i+1,j+1})(z_{i,j+1} - z_{i,j})} = \frac{\alpha_i}{\beta_j}.$$

Theorem (A-dT-M '22)

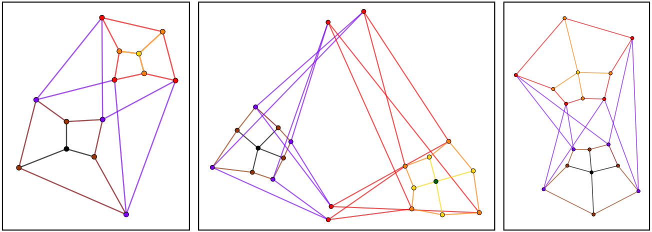
Let $m \geq 1$, and let $\tilde{z}$ be an m-periodic integrable cross-ratio map such that $\tilde{z}_{i,0} = 0$ for all $i \in 2\mathbb{Z}$. Then, if the following are well defined,

$$\forall\, i \in 2\mathbb{Z}, \qquad \tilde{z}_{i,2m-1} = \frac{\sum_{\ell=0}^{m-1} (\alpha_\ell - \beta_\ell)}{\sum_{\ell=0}^{m-1} \frac{1}{\tilde{z}_{2\ell+1,1}} (\alpha_\ell - \beta_{-\ell-1})},$$

where $\tilde{z}$ is z rotated by $-45°$.

Consequences for discrete holomorphic functions

- Specifying $\alpha_\ell \equiv -1$, $\beta_\ell = 1$, we recover a theorem of [Yao '14] when m is odd.
- When m is even, using relation to P-nets, we prove that the singularity occurs one step earlier.

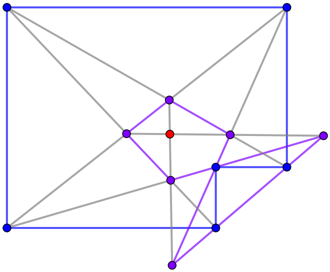


Propagation of discrete holomorphic functions. Left: 3-periodic case. Center and right: 4-periodic case.

Application to pentagram map

Theorem [Schwartz '07, Glick '14, Yao '14]

If $(v_1, \ldots, v_{2m})$ is an axis-aligned polygon, then if well-defined, $T^{m-1}(v_1, \ldots, v_{2m})$ is reduced to a point, which is the center of mass of $(v_1, \ldots, v_{2m})$.

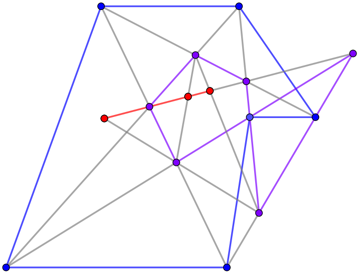


$\rightsquigarrow$ We provide an alternative proof.

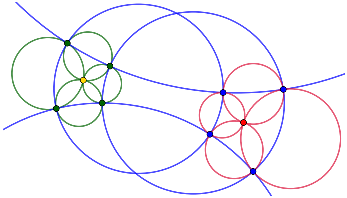
Application to pentagram map

Theorem [A-dT-M '22]

If $(v_1, \ldots, v_{2m})$ is half-axis-aligned, then $T^{2m-4}(v_1, \ldots, v_{2m})$ is singular (that is, $T^{2m-3}(v_1, \ldots, v_{2m})$ is undefined).



Conclusion



System	Initial condition	Steps	Citations
Miquel	m-Dodgson	$m-1$	new
P-nets	m-Dodgson	$m-1$	[Glick15, Yao14]
P-nets	m-Dodgson*	$m-2$	new
Int. cr-maps	$(m,2)$-Devron	$2m-2$	new
D. hol. f., $[m]_2 = 1$	$(m,2)$-Devron	$2m-2$	[Yao14]
D. hol. f., $[m]_2 = 0$	$(m,2)$-Devron	$2m-3$	new
D. hol. f., $[m]_2 = 0$	$(m,2)$-Devron*	$2m-4$	new
Orthogonal CP	$(2m,2)$-Devron	$m-2$	new
Polygon recutting	$(m,1)$-Devron	$m-1$	[Glick15]
Circle intersection dyn.	$(m,1)$-Devron	$m-1$	new
Circle intersection dyn.	$(m,2)$-Devron	$2m-4$	new
Pentagram map	m-Dodgson	$m-1$	[Glick15, Yao14]
Pentagram map	$(m,2)$-Devron	$2m-4$	new