Discrete vector bundles and determinantal processes

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Setup and question

Vector bundles

Discrete vector bundle: a graph with isomorphic vector spaces of same dimension (called the rank) over its vertices (and edges)

Connection: isomorphisms between these fibers (we will consider inner-product spaces and isometries)

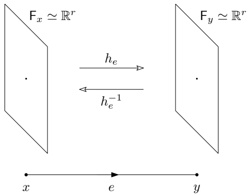
From discretization of continuous vector bundles: graph embedded on the base space of a bundle.

- tangent bundle of round sphere $S^2 \subset \mathbb{R}^3$: transport reflects curvature; example of a geodesic triangle with 3 right-angles, and holonomy $\pi/2$.
- flat case: representation of fundamental group of a manifold
- Moebius strip: cycle-graph with holonomy -1

Discrete setup: confusion between topology and geometry, both encoded in the connection h. Non triviality comes from the cycles in the graph.

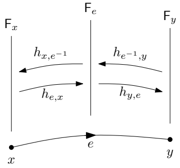
Discrete vector bundles as networks with holonomy

Graph with connection:



Holonomy: composition of isomorphisms along paths

In fact, we will need a refined bundle: adding a fiber over each edge (on a metric graph, we would have a bundle over the segment of that edge)



Notion of gauge: choice, transformation, covariance, invariance

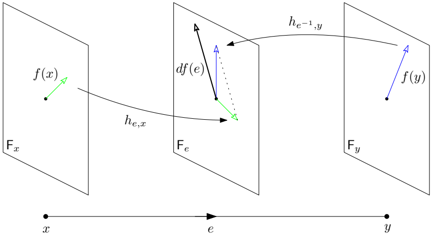
Think of example of sphere: choice of basis of each fiber affects the matrices on edges, but not the holonomies of closed path up to conjugation.

In general, there is an internal degree of freedom which consists in automorphisms of the fibers of the bundle. Conjugating the h by these is called a gauge transformation. Holonomies are conjugated too, so their traces are invariant.

Some quantities are gauge invariant, some covariant.

Covariant derivative

In the talk, we assume the graph to be finite and connected.



covariant derivative d: for this to be thought of as a derivative, we imagine the edge to be infinitesimally small (see later, although on a finite graph).

Laplacian: $\Delta = d^*d$. Dirichlet energy:

$$\mathcal{E}(f) = \frac{1}{2} \sum_{e=xy} \|f_x - h_e^{-1} f_y\|^2 = \langle f, \Delta f \rangle$$

In general, for f of fixed norm, minimum is not zero, unlike when $h = 1$.

Probability on these networks with holonomy?

In the case where $h = 1$, we have the plain geometry of a graph, and we know of interesting probabilistic objects: random walk, Gaussian free field (GFF), loop soups, uniform spanning tree.

When $h \neq 1$ or the rank $N \geq 2$, we still have these objects, but we would like to see more.

Question

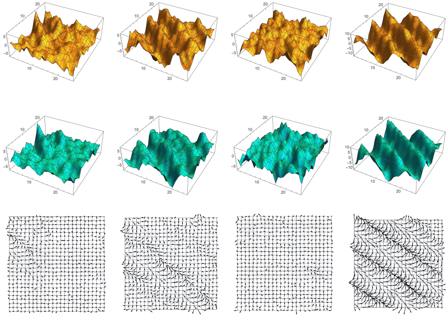
Which objects can we define which would take into account the presence of h and what are the relations between them?

Related but different question: study field of random h (lattice gauge theory). Instead, we fix h (random or not), and study processes in the environment determined by h.

The environment h can be interpreted as 'disorder' variables in a spin system (notion of frustration).

Gaussian free vector field

The Dirichlet action defines a Gaussian measure: a Gaussian free vector field Φ.



Unless h is trivial, it is not determined by N iid scalar GFFs (see sample).

Isomorphism theorems

Link to random walk holonomies [K.–Lévy, 2016]:

If x and y are vertices, and if ξ and η are respectively vectors of the fibers over x and y, then

$$\mathbb{E}[\langle \xi, \Phi_x \rangle \langle \Phi_y, \eta \rangle] = \int \langle \xi, \mathrm{hol}(\gamma)^{-1}(\eta) \rangle \ \mathrm{d}\nu_{x,y}(\gamma),$$

where $\nu_{x,y}$ is a finite measure on the set of paths from x to y.

Isomorphism theorems

Link to loop soups weighted by matrix-coefficients of holonomies of paths.

Let us illustrate on a particular example inspired by Titus Lupu [Lupu, 2019]. Consider the real vector bundle of Hermitian matrices, and connection acts by conjugation: $M_x \mapsto h_{xy} M_x h_{xy}^{-1}$.

Let Ψ be the Gaussian random matrix field (Gaussian Unitary Ensemble, GUE over each vertex, but all correlated).

Theorem ([K.–Lévy, 2016])

The field $x \mapsto \frac{1}{2}\mathrm{Tr}(\Psi_x^* \Psi_x) = \frac{1}{2}\sum_i \lambda_{x,i}^2$ has the same distribution as the occupation field of a Poisson point process of loops with intensity $\frac{1}{2}\mathrm{Tr}(\mathrm{hol}_h(\cdot))^2 \mu$, where μ is the 'usual' loop measure.

Partition function

These identities are proved by computing Laplace transforms of the distributions which involve $\det \Delta$ (depending on some parameters which deform Δ). For instance, the partition function of the Gaussian field is $(\det \Delta)^{-1/2}$.

This is the 'bosonic' story.

How do we find other 'fermionic' processes in there? Is $\det \Delta$ the partition function of some probabilistic model?

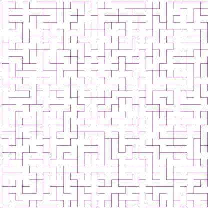
Determinantal point processes

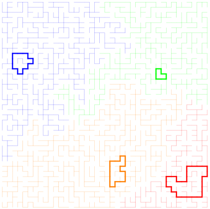
Spanning trees and cycle-rooted spanning forests

Matrix-tree theorem says that a volume (a determinant) is equal to a partition function (a polynomial counting something).

Rank $N = 1$, $h = 1$, Kirchhoff:

$$\det{}' \Delta = \sum_{\text{spanning trees}} \prod_e c_e$$





Rank $N = 1$, $h \neq 1$, Forman–Kenyon:

$$\det \Delta_h = \sum_{\text{CRSF}} \prod_e c_e \prod_{\textit{cycles } c} (2 - 2\Re \mathrm{hol}(c))$$

Trees and forests as determinantal point processes

Burton–Pemantle, and then Kenyon showed that if we pick the above subgraphs with probability proportional to the weight in the partition function, then these are determinantal point processes (DPP).

The local correlations are given by

$$\mathbb{P}(\text{edges } e_1, \ldots, e_k \text{ are occupied}) = \det[(K_{e_i,e_j})_{1 \le i,j \le k}]$$

where K is the matrix in an orthonormal basis indexed by edges, of the orthogonal projection on $\mathrm{im}(d)$, the range of the (covariant) derivative in the space Ω^1 of 1-forms.

General rank N

Start with case of diagonal connection: independence, partition function is product, etc. but we can look for something more general.

We have formulas for partition function but not clear how to interpret (see below however).

So we will generalize the DPP point of view.

Random vector subspaces

DPP on a finite set is a probability measure on subsets X such that there is a matrix K, such that for any subset S, $\mathbb{P}(S \subset X) = \det K_S^S$.

Geometric point of view on DPP: determinantal linear processes (DLP). A kernel is the matrix of an operator k in a basis. This choice of basis is important: think of taking a basis which diagonalizes k (DPP becomes Poisson).

In our case, we have a vector space Ω^1 given by a splitting rather than coming with a particular basis.

$$\Omega^1 = \oplus_e \mathsf{F}_e$$

with $\mathsf{F}_e \simeq \mathbb{R}^N$.

Random vector subspaces

Setup: inner product vector space E of dimension d split as the orthogonal direct sum $E = E_1 \oplus \ldots \oplus E_s$. We denote by σ this splitting.

Given a kernel k we define a way to pick a random subspace Q built from the building blocks E_i.

For that matter, let us introduce the notation $\mathrm{Gr}(E)$ for the Grassmannian, i.e. the set of all subspaces. We endow it with the unique invariant finite positive measure ν with total mass 2^d and assigning mass $\binom{d}{n}$ to each $\mathrm{Gr}_n(E)$. It is a 'continuous counting' measure.

Let $\mathrm{Gr}(E, \sigma) \simeq \mathrm{Gr}(E_1) \times \cdots \times \mathrm{Gr}(E_s)$ be the subset of the Grassmannian determined by the splitting σ. There is a unique invariant positive measure ν^σ on this set assigning total mass 2^d.

Density of determinantal measures

Let $0 \leq \mathrm{k} = \mathrm{k}^* \leq 1$ be a self-dual positive contraction; we will call it a kernel.

If Q is a subspace, let Π^Q denote the orthogonal projection on Q.

Then

$$Q \mapsto f_k(Q) = \det(\mathrm{k}\Pi^Q + (1 - \mathrm{k})\Pi^{Q^\perp})$$

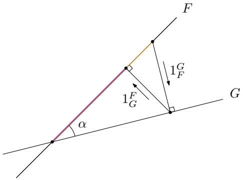
is a positive function on $\mathrm{Gr}(E)$ and

$$\int f_k(Q) d\nu(Q) = 1.$$

Density of determinantal measure

Plücker embedding: $\iota : \mathsf{Gr}(E) \to \mathsf{Gr}_1(\bigwedge E)$,
$H = \mathrm{Vect}(u_1, \ldots, u_k) \mapsto \mathbb{R}(u_1 \wedge \ldots \wedge u_k)$.

In case k is the orthogonal projection on a subspace H of dimension n, then Q is a.s. of dimension n and the density is cosine squared of angle between lines in Plücker embedding of H and Q:



(it is the determinant of the map obtained by composing two projections on figure)

A unique density, different determinantal measures.

Let $\mathrm{Gr}(E,\sigma) \simeq \mathrm{Gr}(E_1) \times \cdots \times \mathrm{Gr}(E_s)$ be the subset of the Grassmannian determined by the splitting σ. There is a unique invariant positive measure ν^σ on this set assigning total mass 2^d.

For any splitting σ,

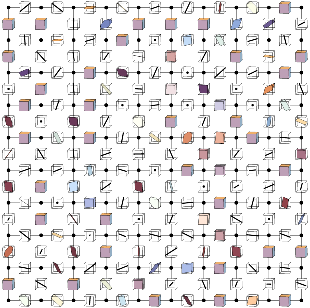
$$\int_{\mathrm{Gr}(E,\sigma)} f_k(Q) d\nu^\sigma(Q) = 1$$

Interpretation: these measures come from different observables of the same state (see later).

Way to sample: pick a uniform random basis in each E_i, then sample a DPP with kernel matrix in that basis, and look at the associated random subspace.

Example: Grassmannian-valued field

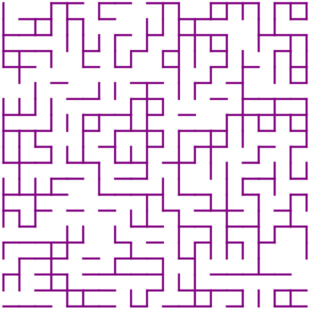
DLP with kernel given by orthogonal projection on $\mathrm{im}(d_h)$.



Here, connection is chosen according to $SO(3)$ Yang–Mills measure on the plane.

Example: dependent percolation field

With a choice of basis $(u_e^i)_{e,i}$. If we look at the DPP induced on $(u_e^1)_e$.



Example: in the trivial case of $\mathbb{R}^N$, get a kernel $K^u_{e,e'} = K_{e,e'}\langle u_e, u_{e'}\rangle$ where K is the UST kernel.

Projection formula

When $\mathsf{k} = \Pi^H$, the random subspace Q is an 'approximation' to H given by the splitting σ:

$$E = \mathsf{Q} \oplus H^\perp.$$

Moreover

$$\mathbb{E}[P^{\mathsf{Q}}_{\| H^\perp}] = \Pi^H$$

In fact more:

Theorem ([K.–Lévy, 2016])

$$\mathbb{E}[\bigwedge P^{\mathsf{Q}}_{\| H^\perp}] = \bigwedge \Pi^H$$

(Part of this theorem or special cases were known to various authors.)

Recall: $\bigwedge a(e_1 \wedge \ldots \wedge e_k) = ae_1 \wedge \ldots \wedge ae_k$.

Kirchhoff's electrical formula

Now let us turn back to our model. We can rewrite the projection formula as

$$\mathbb{E}[P^{\lozenge}_{\|\mathsf{Q}}] = \Pi^{\lozenge}$$

In the case of rank 1 and $h = 1$, Q is a uniform spanning tree. This formula allows to write a current as an average current flowing along branches of UST (Kirchhoff's formula).

Interpretation

The point of view of 'several observables for one common state' can be stated using the language of positive operator-valued measures.
Define an observable: the map which to a Borel set B associates the operator

$$O_\sigma(B) = \int_B \Pi^{\iota(Q)} \,\mathrm{d}\nu^\sigma(Q).$$

To every kernel k on E corresponds a density of states $\rho_{\mathsf{k}} \in \mathrm{End}(\bigwedge E)$.
When k is orthogonal projection onto H, ρ_{k} is orthogonal projection onto $\iota(H)$. When $\mathsf{k} < 1$,

$$\rho_{\mathsf{k}} = \det(1-\mathsf{k}) \bigwedge (\mathsf{k}(1-\mathsf{k})^{-1}).$$

Theorem ([K.–Lévy, 2016])

Let Q *be a determinantal linear process of* (E,σ) *with kernel* k. *Then for every Borel subset* B *of* $\mathsf{Gr}(E,\sigma)$, *one has the equality*

$$\mathbb{P}(\mathsf{Q} \in B) = \mathrm{Tr}_{\bigwedge E}(O_\sigma(B)\rho_{\mathsf{k}})\,.$$

Partition function

Let us come back to the partition function $\det \Delta$. We have obtained several matrix-tree type formulas but not easy to interpret.

Most gauge invariant one:

Theorem (K.–Lévy, 2018, unpublished yet)

$$\det \Delta = \sum_{N-CRSF} \prod_{e} c_e \sum_{\substack{\text{multiset } S \\ \text{of cycles}}} \frac{\prod_{c \in S} -\mathrm{Tr}(\mathrm{hol}(c))}{S! \prod_{c \in S} \mathrm{val}(c)}$$

Allows to recover the formula of Forman in rank $N = 1$.

Another way to obtain formulas is to use the DPP framework: in general, we find an integral of an explicit matrix, but which is not very combinatorially-explicit.

Partition function

Let us give a somewhat 'nicer' expression for $h = e^{tA}$ close to the identity ($t \to 0$), which has a geometric meaning too (scaling limit, although here graph is finite):

Theorem ([K.–Lévy, 2021])

$$\lim_{t\to 0} t^{-2N} \det \Delta_t = \sum_{\underline{T}=(T_1,\ldots,T_N)} \left\langle x_A, \left(\bigwedge^N \mathrm{P}^{\Diamond}_{\|Q_{\underline{T}}}\right) x_A \right\rangle_{\bigwedge^N \Omega^1}.$$

where $Q_{\underline{T}}$ is the space spanned by the N spanning trees T_i seen as subspaces and $x_A = A_1 \wedge A_2 \wedge \ldots \wedge A_N$.

The way we prove it is using the projection theorem stated before.
The underlying probability measure is a DPP with projection kernel on the space $\bigstar_A = \mathrm{im}(d) \oplus \mathrm{Vect}(A_1, \ldots, A_N)$.
In rank 1, the right-hand side becomes
$\sum_{\text{unicycles with cycle c}} A_c^2$, where A_c is the holonomy angle of the unique cycle (a formula easily obtained from the Forman–Kenyon theorem, and considered by Kenyon).

Perspectives

Perspectives

Infinite graphs: well-defined objects, analogs of wired and free uniform spanning forests. In the case of periodic connections: generically, exponential decay of correlations. However, other cases to study. PhD project of Héloïse Constantin is to try to exhibit critical measures for these spanning forests and describe the phase diagram.

Large N limit (easy remark about independence of percolation model defined above)

Metric graphs; Continuous analogs

The results seem to extend to DPP with non-symmetric kernels: does the projection formula say something interesting for dimers (e.g. analog of Kirchhoff's electrical result, maybe well-known)?

Random connection: weighted by partition function (analogous to random map geometry deformations)

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Thank you for your attention