Universality of spin correlations in the Ising model on isoradial graphs.

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w/ Dmitry Chelkak (ENS) and Konstantin Izyurov (Helsinki) arXiv: 2104.12858
Jhih-Huang Li (Warwick) arXiv:
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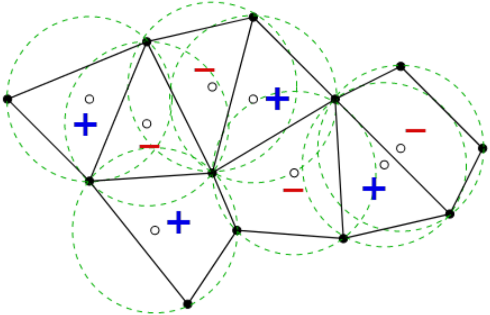
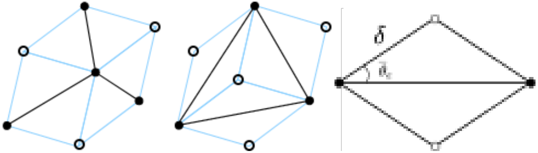
ÉCOLE NORMALE SUPÉRIEURE

Z-invariant model

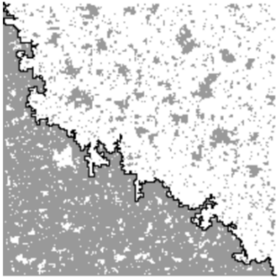
- $$\mathbb{P}_{\mathcal{G},J}(\sigma) \propto \exp\Big(\sum_{e=uv\in\mathcal{G}} J_e \sigma_u \sigma_v\Big)$$
- $Y-\Delta$ invariance, Yang-Baxter equations
- $$J_e := J(\overline{\theta}_e|k) = \frac{1}{2}\log\left[\frac{1+\mathrm{sn}(\frac{2K(k)}{\pi}\overline{\theta}_e|k)}{\mathrm{cn}(\frac{2K(k)}{\pi}\overline{\theta}_e|k)}\right]$$
- Weights $\Longrightarrow$ Angles + $k^2 \in]-\infty, 1]$
- Second order phase transition at $k=0$ (Boutillier, de Tilière, Raschel, 2019)
- Positive magnetization and exponential decay (Lis, 14)
- Angles bounded away from 0 and π

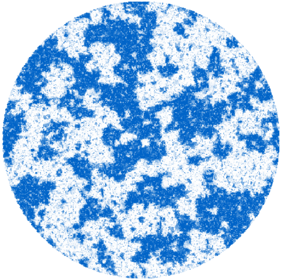




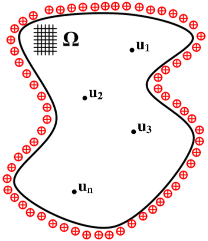
Conformally invariant scaling limit at criticality ($k = 0$)

- Dobrushin interface to **SLE**(3) (Smirnov et al, 2006+)
- Full loop ensemble to **CLE**(3) (Benoist, Hongler, 2019)





- Energy density (Hongler, Smirnov, 2013) $(\delta \sin(\theta_e))^{-1}(\mathbb{E}_{\Omega_\delta}[\varepsilon_e] - \mathbb{E}_{\mathbb{C}_\delta}[\varepsilon_e]) \underset{\delta \to 0}{\longrightarrow} \frac{1}{2\pi} l_\Omega(a)$
- Spins (Chelkak, Hongler, Izyurov, 2015) $\delta^{-\frac{n}{8}}\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_1} \cdots \sigma_{u_n}] \underset{\delta \to 0}{\longrightarrow} \mathcal{C}_\sigma^n \langle \sigma_{u_1} \cdots \sigma_{u_n} \rangle^{+}_{\Omega}$
- $\mathcal{C}_\sigma = 2^{1/6} e^{\frac{3}{2}\zeta'(-1)}$ **a priori** lattice dependant.



Massive scaling limit ($q = \frac{1}{2}m\delta$)

- Scale temperature & lattice $\mathcal{G}_\delta$
- $q := e^{-\pi\frac{K'(k)}{K(k)}} = \frac{1}{2}m\delta \sim \frac{k^2}{16}$
- Krammers-Wannier duality : $q \longleftrightarrow -q$
- Sub-criticality $q > 0$
- Spins $m < 0$, smooth $\partial\Omega$ (Park, 18)

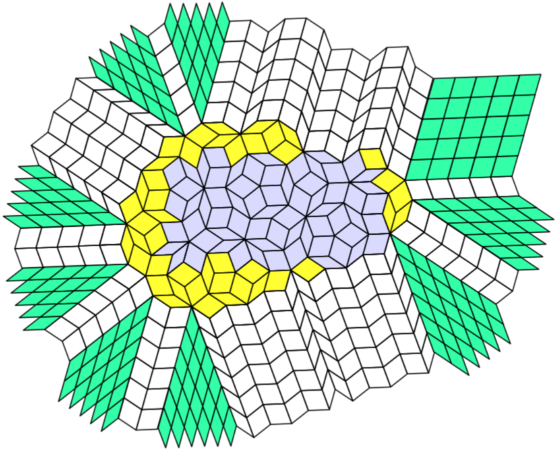
$$\delta^{-\frac{n}{8}}\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{u_1}\cdots\sigma_{u_n}\right] \xrightarrow[\delta\to 0]{} \mathcal{C}^n_\sigma\langle\sigma_{u_1}\cdots\sigma_{u_n}\rangle^m_\Omega$$

Main results

- Full-plane massive scaling limit m both signs (Chelkak, Izyurov, M.)
$$\delta^{-\frac{1}{4}}\mathbb{E}^{(m)}_{\mathcal{G}_\delta}[\sigma_{u_1}\sigma_{u_2}] \xrightarrow[\delta\to 0]{} \mathcal{C}^2_\sigma \cdot \Xi(|u_2 - u_1|, m)$$
- Universality of $\mathcal{C}_\sigma$ among isoradial graphs
- Rotational invariance of Ξ, $\Xi(r, 0) = r^{-\frac{1}{4}}$ and $\Xi(r, m) \underset{r\to 0}{\sim} r^{-\frac{1}{4}}$
- When $m \leq 0$, universality of
$$\delta^{-\frac{n}{8}}\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_{u_1}\cdots\sigma_{u_n}\right] \xrightarrow[\delta\to 0]{} \mathcal{C}^n_\sigma\langle\sigma_{u_1}\cdots\sigma_{u_n}\rangle^m_\Omega$$
- New techniques applicable beyond isoradial setup

Universality outside of criticality

- Baxter formula $\mathbb{E}^{+}_{\mathcal{G}_1^\circ}[\sigma_0] = (k^\star)^{\frac{1}{4}}$.
- $\mathbb{E}^{+}_{\mathcal{G}_1^\circ}[\sigma_u \sigma_v] = (k^\star)^{\frac{1}{2}}$ as $|u - v| \to \infty$.
- Universal limits, not computed explicitly.
- Formula on rectangular-lattice (modern proof in arXiv : 1904.09168).
- Extending large pieces of isoradial grids. Train track convexity (de Tilière 07).

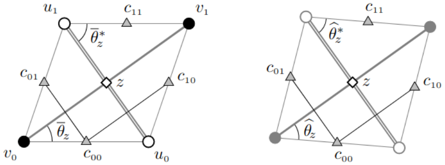


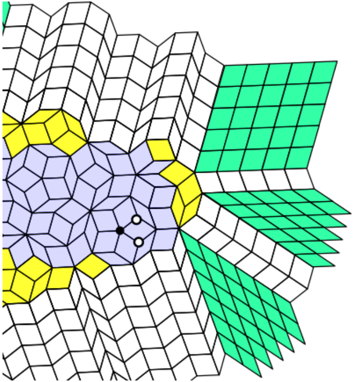
A simple proof of the Baxter formula

- Kadanoff-Ceva $X_\infty(c) := \langle \mu_c \sigma_c \mu_{v^\bullet} \sigma_\infty \rangle_{\mathcal{G}}$

$$\begin{cases} X_\infty(c_{00}) = \cos(\hat{\theta}_z^*) X_\infty(c_{01}) + \sin(\hat{\theta}_z^*) X_\infty(c_{10}) \\ \text{Decay at } \infty \\ \text{Branching around } v^\bullet \end{cases}$$

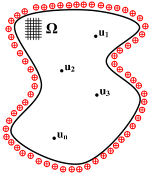
- $\bar{\theta}_z^* - \hat{\theta}_z^* = 2m\delta + O(\delta^2)$ in the massive-case
- $X_\infty \propto G_{[v^\bullet]}$ constructed explicitly via massive discrete exponentials
- $\mathbb{E}_{\mathcal{G}}^+[\sigma_{u_1}] \overset{KC}{=} X_\infty(v^\bullet u_1) \overset{G_{[v^\bullet]}}{=} X_\infty(v^\bullet u_2) \overset{KC}{=} \mathbb{E}_{\mathcal{G}}^+[\sigma_{u_2}]$
- Universality on one grid + RSW (Park, 21) $\implies$ Universality on all grids





Core of the proof

Sharp control of log derivative $$\boxed{\frac{\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_1}\sigma_{u_2}]}{\mathbb{E}^{+}_{\Omega_\delta}[\sigma_{u_1'}\sigma_{u_2}]} = 1 + \Re\left[\mathcal{A}^{(m)}_{(\Omega,u_1,u_2)}(u_1-u_1')\right] + o(\delta)}$$

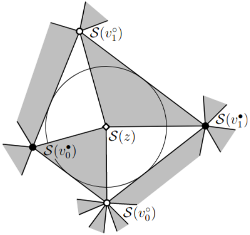


- Abstract discrete Cauchy formula
  $$\boxed{\mathbb{E}_{\Omega_\delta}[\sigma_{u_1}\sigma_{u_2}] = \oint F^{\diamond}_{KC}(z)\cdot G_{[u]}(z)dz}$$
  (similar to $F(a) = \oint F(z)\cdot K(a,z)dz$)
- Convergence of discrete $F^{\diamond}_{KC}$
- Construction of $G_{[u]}$ via discrete exponentials $e^{\pm 2m|z|}z^{-\frac{1}{2}}$

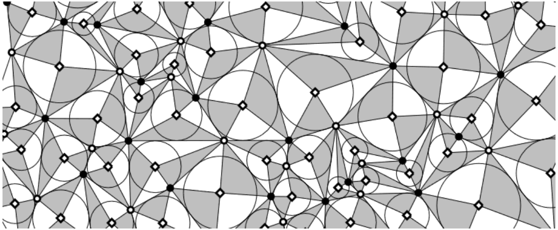
- Definition of continuous correlations
- $\langle \sigma_{u_1}\sigma_{u_2}\rangle^{+}_{\Omega} := \exp[\int^{(u_1,u_2)} \mathcal{A}^{(m)}_{(\Omega,w,s)}]$
- Universality in critical/massive full plane $\Longrightarrow$ normalizing factor
- Expression in terms of Painlevé IV (McCoy, Wu, Tracy, Barouch, 76)

A geometrical interpretation on $\mathcal{S}$-embeddings

- Ising weights $\overset{Draw}{\Longrightarrow}$ $\mathcal{S}$-embedding
- Massive discrete exponentials solution to propagation equation, $x_e := \tan(\frac{\hat{\theta}_e}{2})$
- Proper embedding (regularity in discrete)
- Origami map $\mathcal{Q}^\delta(v^\bullet) - \mathcal{Q}^\delta(u^\circ) = \delta_{(v^\bullet u^\circ)}$



- Weierstrass parametrization of $z \mapsto (\mathcal{S}(z), \mathcal{Q}(z)) \in \mathbb{R}^{2+1}$
- $m = \frac{1}{2}H\ell$, H mean curvature, ℓ metric element.
- Surface a mean curvature $\propto m$
- General phenomenon arXiv: 2006.14559, section 2.7

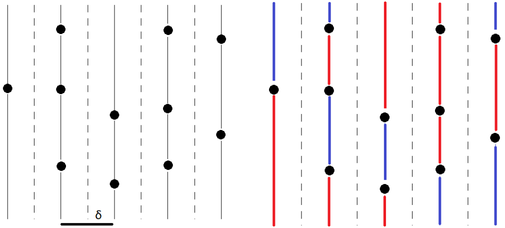


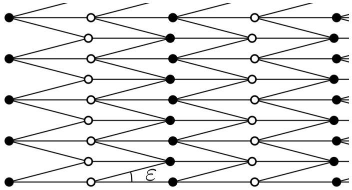
The Quantum Ising model

- I.I.D $\mathcal{PPP}^\bullet(2\delta^{-1})$ on $\delta\mathbb{Z} \times \mathbb{R}$
- Spin measure by Radon-Nikodym :

$$\frac{d\mathbb{P}_{spin}(\sigma)}{dPPP_{\tau_\delta}(D)} \propto \exp(2\delta^{-1} \int_{\Omega^\circ} \varepsilon_e de)$$

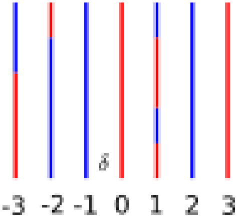
- Criticality (Duminil-Copin, Manolescu, Li, 18)
- Limit of flattened isoradial grids $\varepsilon \to 0$
- Exchange limits $\delta, \varepsilon \to 0$





Scaling limit and 1D space-time formulae

- Interfaces to SLE(16/3) (Li, 18)
- Energy density (Li, M, 21)
- Spins (Li, M, 21)
- Rotational inv, Magnetization, correlation lenght (Li, M, 21)
- Physical interpretation of space-time 1D quantum quantum spin chain QI on $\delta\mathbb{Z}$



- $$\delta^{-1}\mathbb{E}^{+}_{\mathrm{QI}}[\varepsilon_x^{(t)}] \xrightarrow[\delta\to 0]{} \frac{1}{4\pi t}$$
  $$\delta^{-1}\mathbb{E}^{\mathrm{free}}_{\mathrm{QI}}[\varepsilon_x^{(t)}] \xrightarrow[\delta\to 0]{} -\frac{1}{4\pi t}$$
-

$$\delta^{-\frac{1}{4}}\mathbb{E}^{+}_{\delta}[\sigma_{x_1}^{(t_1)}\sigma_{x_2}^{(t_2)}] \xrightarrow[\delta\to 0]{} \mathcal{C}^2\left(\frac{1}{t_1 t_2}\right)^{\frac{1}{8}} \left[\left|\frac{(x_1-x_2)^2+(t_1+t_2)^2}{(x_1-x_2)^2+(t_1-t_2)^2}\right|^{\frac{1}{4}} + \left|\frac{(x_1-x_2)^2+(t_1-t_2)^2}{(x_1-x_2)^2+(t_1+t_2)^2}\right|^{\frac{1}{4}}\right]^{\frac{1}{2}}$$

$$\delta^{-\frac{1}{4}}\mathbb{E}^{\mathrm{free}}_{\delta}[\sigma_{x_1}^{(t_1)}\sigma_{x_2}^{(t_2)}] \xrightarrow[\delta\to 0]{} \mathcal{C}^2\left(\frac{1}{t_1 t_2}\right)^{\frac{1}{8}} \left[\left|\frac{(x_1-x_2)^2+(t_1+t_2)^2}{(x_1-x_2)^2+(t_1-t_2)^2}\right|^{\frac{1}{4}} + \left|\frac{(x_1-x_2)^2+(t_1-t_2)^2}{(x_1-x_2)^2+(t_1+t_2)^2}\right|^{\frac{1}{4}}\right]^{\frac{1}{2}}$$