Ising model, total positivity, and criticality

Pavel Galashin (UCLA)

Séminaire DIMERS
June 30, 2021

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[Gal21] P. Galashin. Critical varieties in the Grassmannian. arXiv:2102.13339.

Ising model

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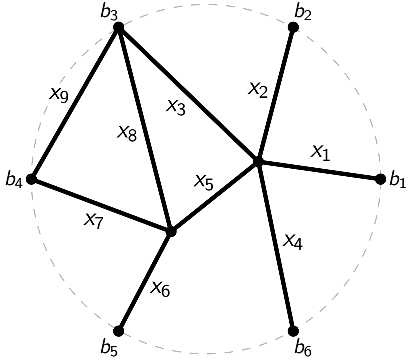

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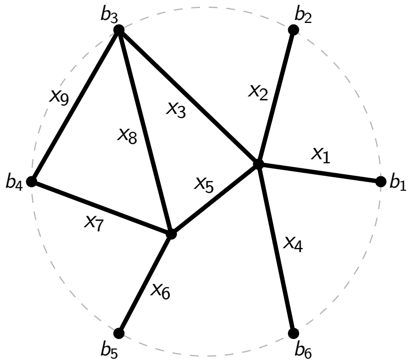

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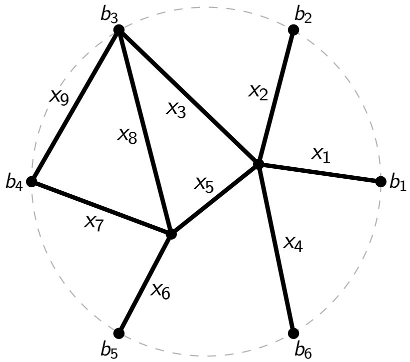

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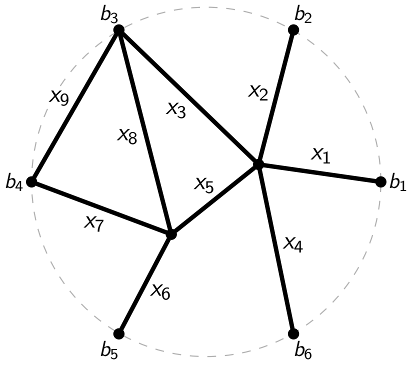

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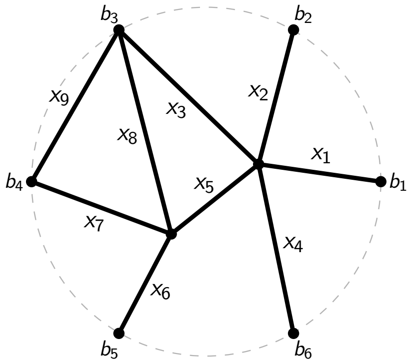

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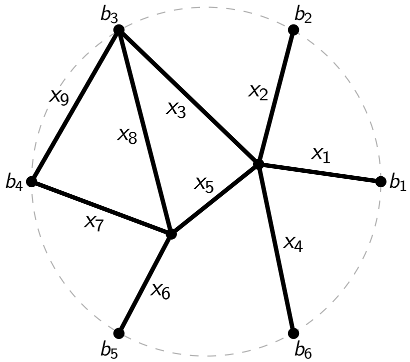

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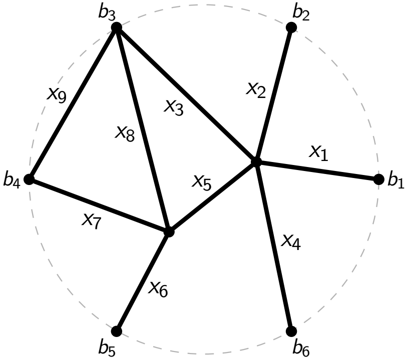

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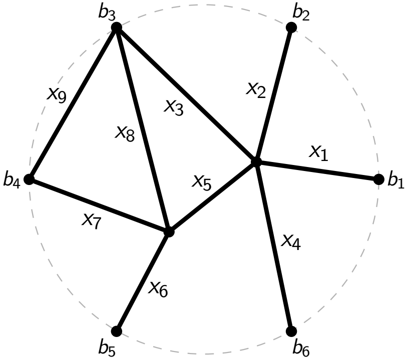

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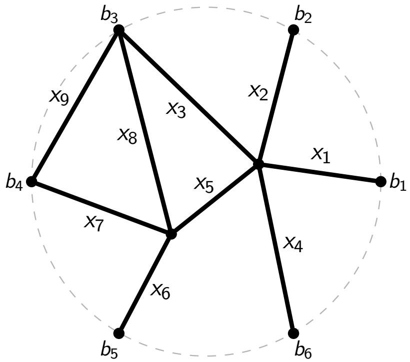
Q2: How to reconstruct the edge weights from boundary correlations?



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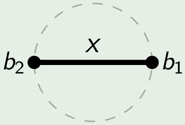

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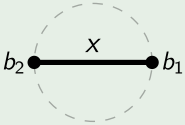

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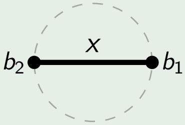

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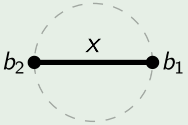

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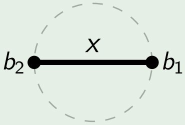

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Total positivity

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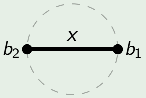

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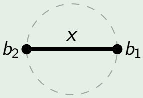

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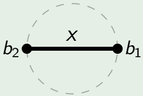

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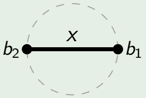

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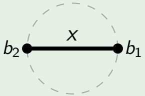

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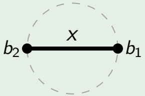

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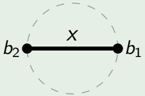

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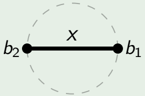

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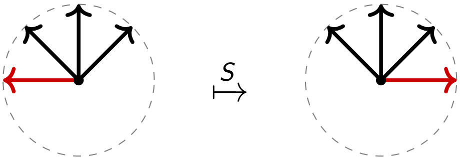

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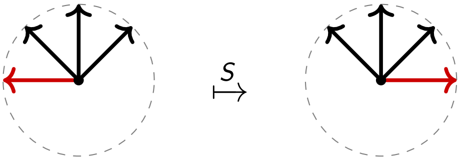

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Theorem (G.–Karp–Lam (2017))

$\mathrm{Gr}_{\geqslant 0}(k,n)$ is homeomorphic to a $k(n-k)$-dimensional closed ball.

Our proof involves a flow that contracts the whole $\mathrm{Gr}_{\geqslant 0}(k,n)$ to the unique cyclically symmetric point $X_0 \in \mathrm{Gr}_{\geqslant 0}(k,n)$.

Cyclic shift $S : \mathrm{Gr}(k,n) \to \mathrm{Gr}(k,n), \quad [w_1|w_2|\dots|w_n] \mapsto [(-1)^{k-1}w_n|w_1|\dots|w_{n-1}]$.

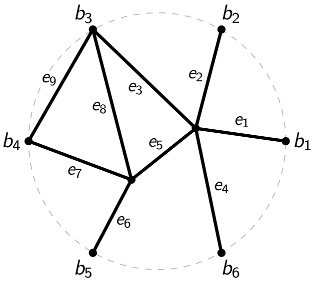
This map preserves $\mathrm{Gr}_{\geqslant 0}(k,n)$.

Example: For $\mathrm{Gr}_{\geqslant 0}(2,4)$, we have

$$X_0 = \begin{pmatrix} 1 & 0 & -1 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 & 0 \end{pmatrix} \overset{S}{\mapsto} \begin{pmatrix} \sqrt{2} & 1 & 0 & -1 \\ 0 & 1 & \sqrt{2} & 1 \end{pmatrix} = X_0 \in \mathrm{OG}_{\geqslant 0}(2,4)$$

$$\begin{aligned} &\Delta_{13} = 2 \quad \Delta_{12} = \sqrt{2} \quad \Delta_{14} = \sqrt{2} \\ &\Delta_{24} = 2 \quad \Delta_{34} = \sqrt{2} \quad \Delta_{23} = \sqrt{2}. \end{aligned}$$

Kramers–Wannier's duality



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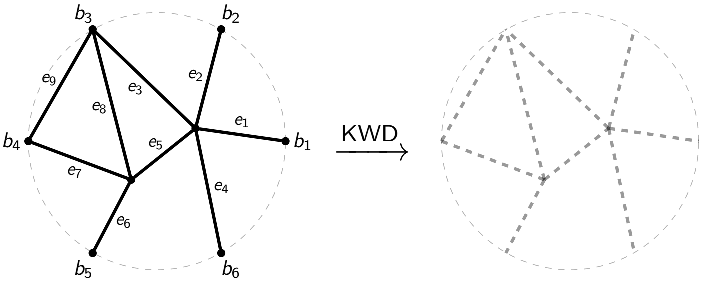

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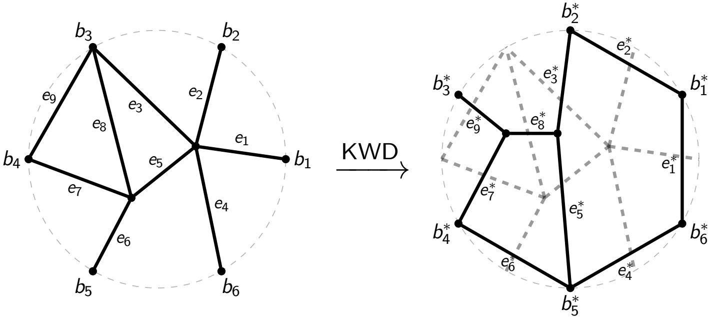

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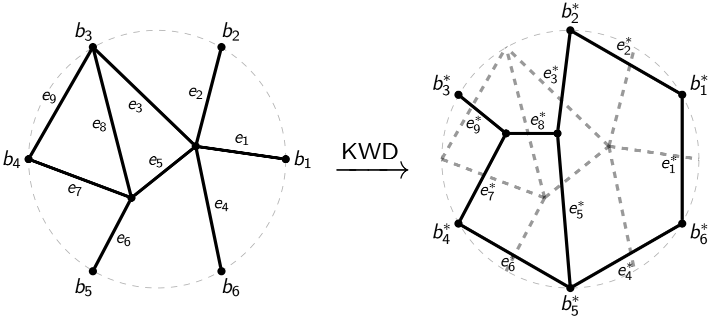

Q: what happens if we apply the duality twice?

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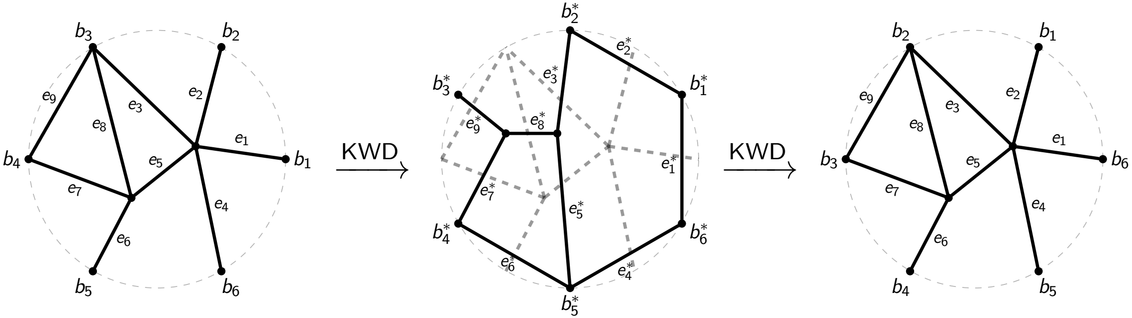

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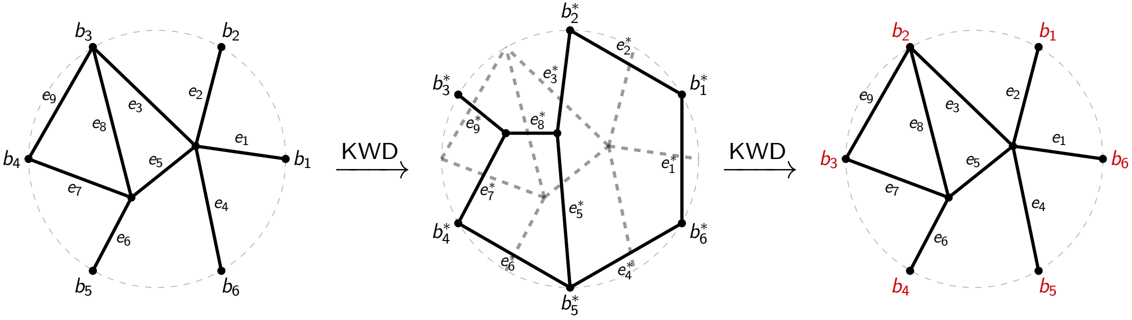

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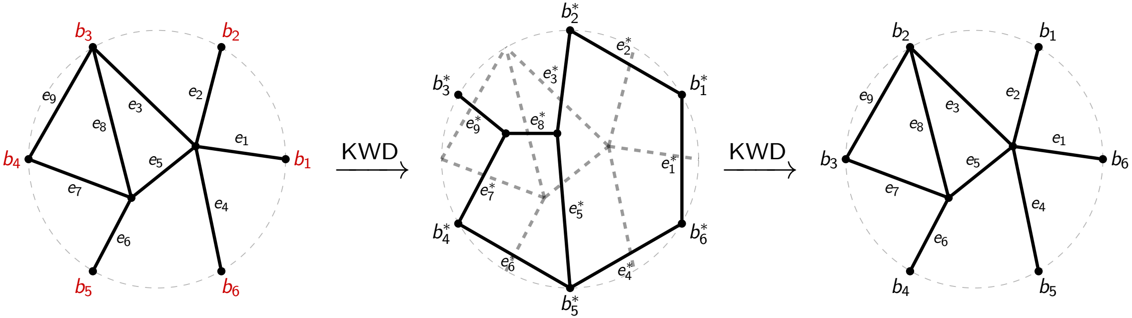

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$$(\mathrm{KWD})^{2n} = \mathrm{id}.$$

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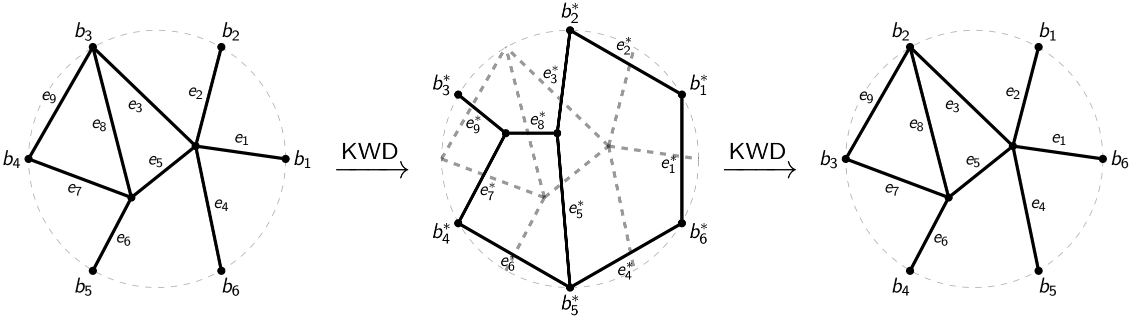

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Doubling map takes KWD to the cyclic shift on $\mathrm{OG}_{\geqslant 0}(n, 2n)$.

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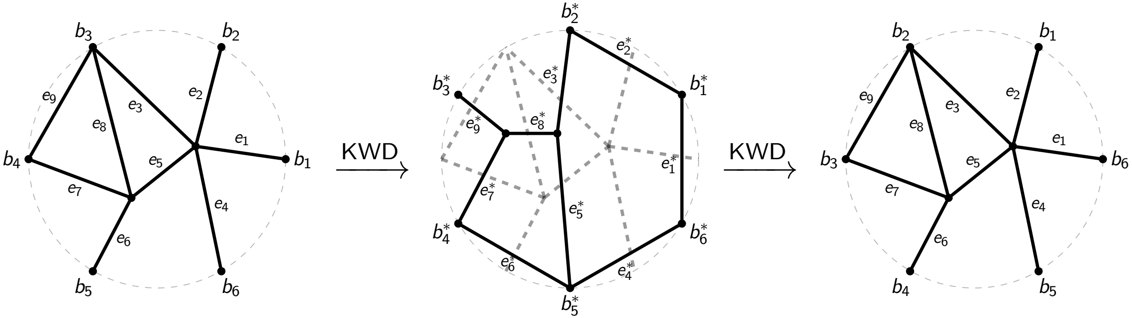

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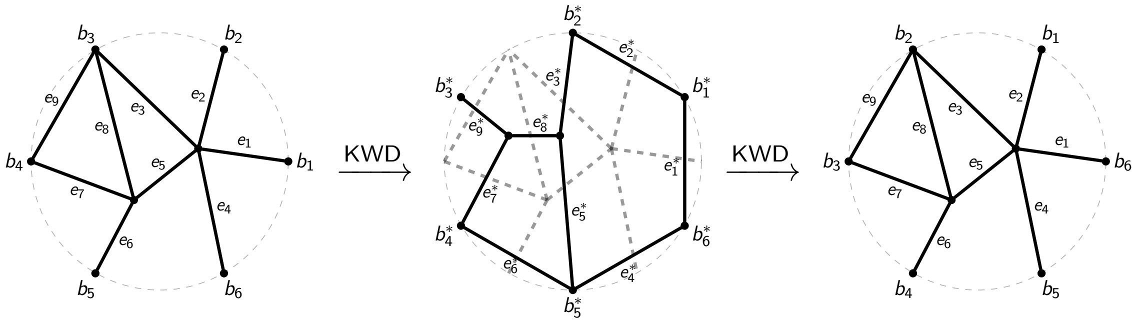

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Critical Ising model $\xleftrightarrow{???}$ fixed point of KWD $\leftrightarrow$ fixed point of cyclic shift.

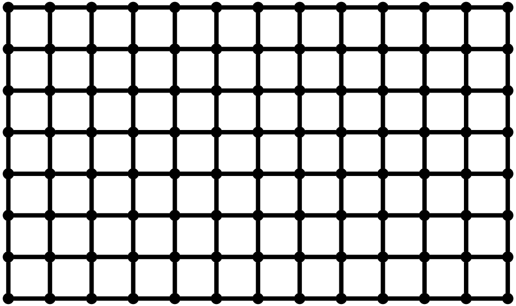
Critical Ising model

Phase transition

$$\mathrm{Prob}(\sigma) := \frac{1}{Z} \prod_{\substack{\{u,v\}\in E(G):\\ \sigma_u \neq \sigma_v}} x_{\{u,v\}}.$$

Usually:

- G = large piece of a (e.g. square) lattice;
- $x_e = x$ for all $e \in E(G)$.



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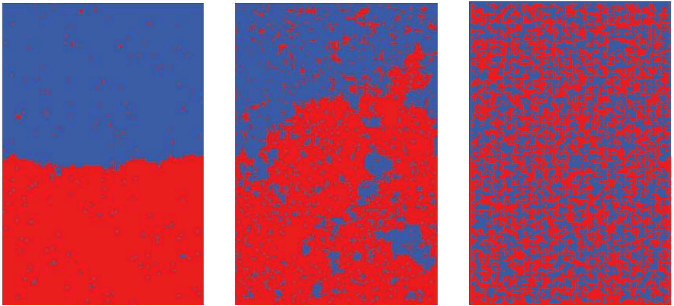

$x < x_{\mathrm{crit}}$ $\quad$ $x = x_{\mathrm{crit}}$ $\quad$ $x > x_{\mathrm{crit}}$

Picture credit: Dmitry Chelkak

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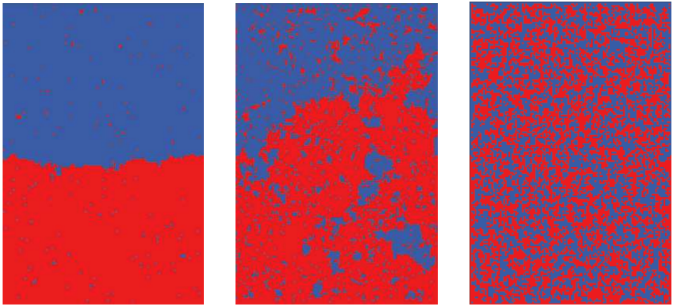
- Square lattice: $x_{\text{crit}} = \sqrt{2} - 1$.



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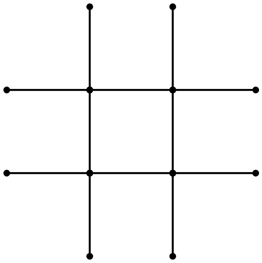

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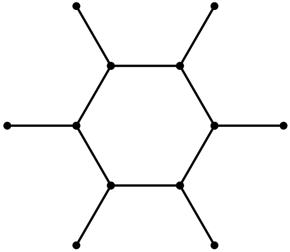
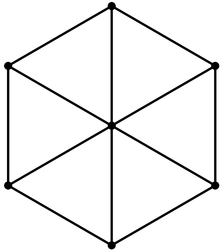


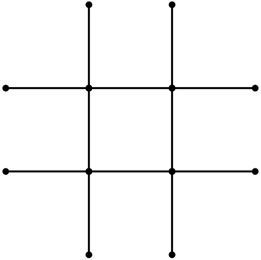
Hexagonal lattice

$x_{\text{crit}} = 2 - \sqrt{3}$



Triangular lattice

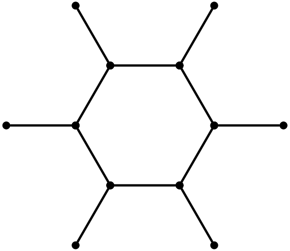
$x_{\text{crit}} = \frac{1}{\sqrt{3}}$



Square lattice

$x_{\text{crit}} = \sqrt{2} - 1$

$x_{\text{crit}} = \tan(\pi/8)$



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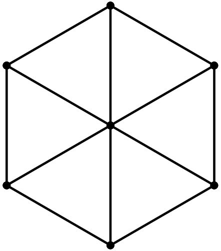

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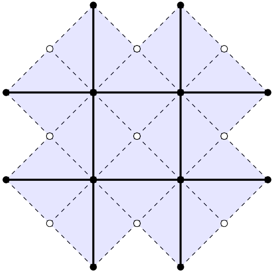

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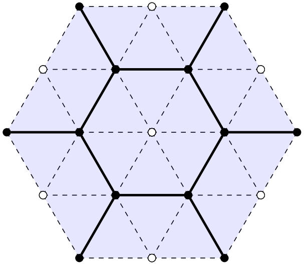

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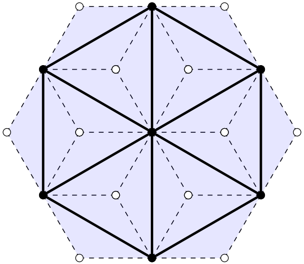

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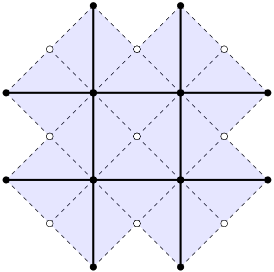

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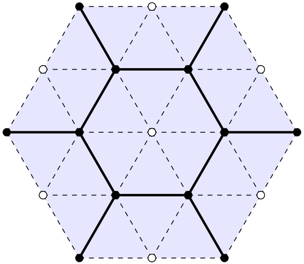

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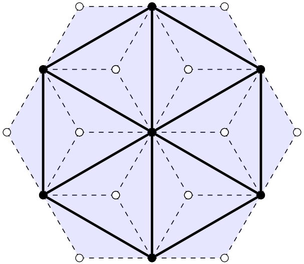

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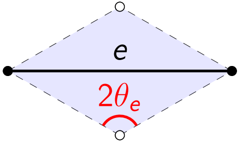

$x_e = \tan(\theta_e/2)$

Critical Z-invariant Ising model

[Bax86] R. J. Baxter. Free-fermion, checkerboard and **Z**-invariant lattice models in statistical mechanics. *Proc. Roy. Soc. London Ser. A*, 404(1826):1–33, 1986.

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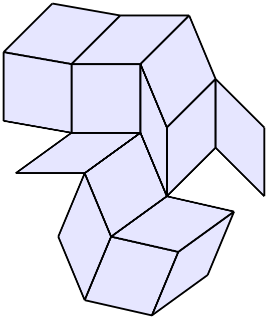

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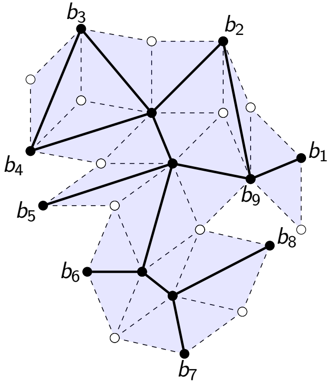

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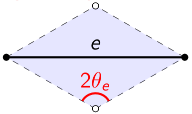

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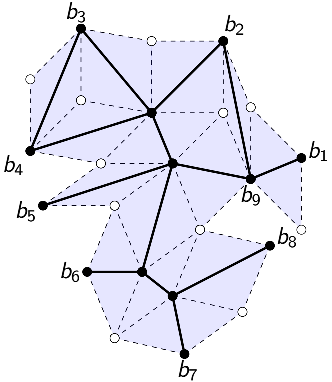

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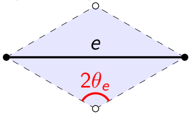

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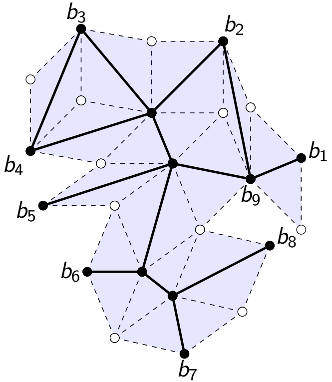

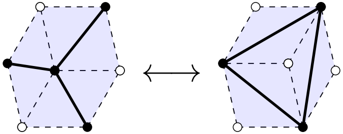

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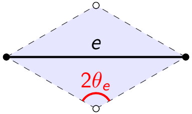

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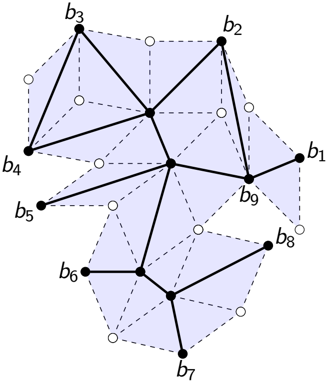
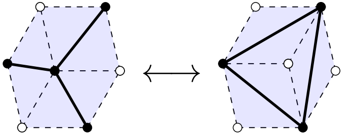
- Conclusion: $\langle \sigma_i \sigma_j \rangle_R$ depends only on the polygonal region R.





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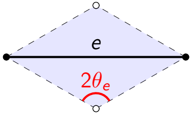

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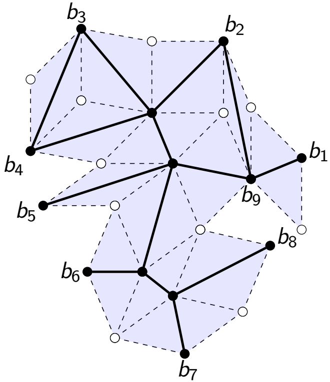
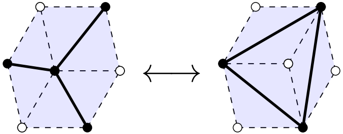
- Formula for $\langle \sigma_i \sigma_j \rangle_R$ in terms of R?

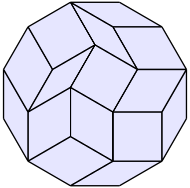


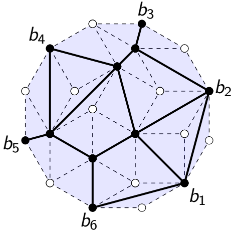


$\longleftrightarrow$

A formula for regular polygons

Let R_N be a regular $2N$-gon and $\langle \sigma_i \sigma_j \rangle_{R_N}$ be the corresponding boundary correlations.





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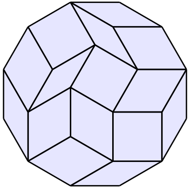

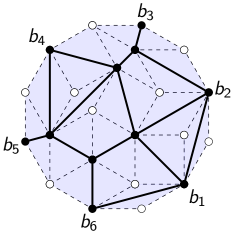

Theorem (G. (2020))

For $1 \leqslant i, j \leqslant N$ and $d := |i-j|$, we have

$$\langle \sigma_i \sigma_j \rangle_{R_N} = \frac{2}{N}\left(\frac{1}{\sin\left((2d-1)\pi/2N\right)} - \frac{1}{\sin\left((2d-3)\pi/2N\right)} + \cdots \pm \frac{1}{\sin\left(\pi/2N\right)} \right) \mp 1.$$

[Gal20] Pavel Galashin. A formula for boundary correlations of the critical Ising model. arXiv:2010.13345, 2020.

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- A: Yes, by the Leibniz formula for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

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[Hon10] Clement Hongler. *Conformal invariance of Ising model correlations.* PhD thesis, 06/28 2010.

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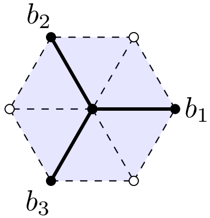
- Treat each edge of G as a resistor.



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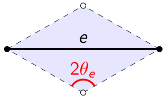

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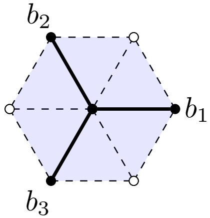

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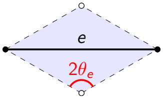

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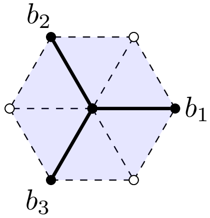

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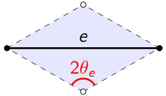

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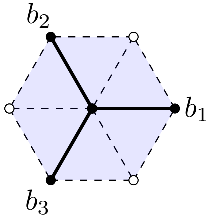

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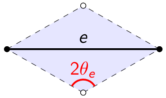

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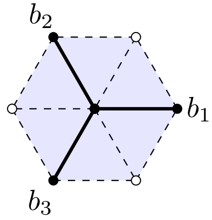

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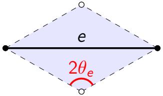

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- Ising model case: $x_e = \tan(\theta_e/2)$ and

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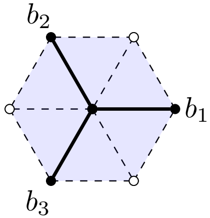

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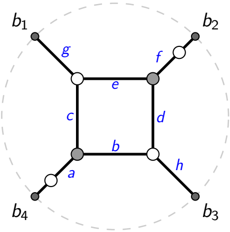
Critical dimer model

- (G, wt) – a weighted planar bipartite graph, with n black boundary vertices $b_1, b_2, \dots, b_n$ of degree 1.



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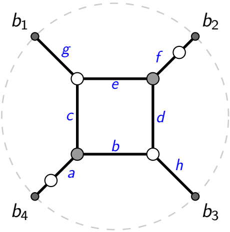

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$$\Delta_J(G, \mathrm{wt}) := \sum_{\mathcal{A}:\partial(\mathcal{A})=J} \mathrm{wt}(\mathcal{A}), \quad \text{where} \quad \mathrm{wt}(\mathcal{A}) := \prod_{e \in \mathcal{A}} \mathrm{wt}(e).$$

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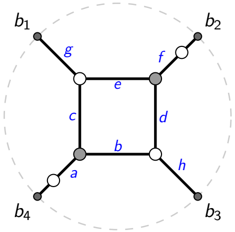

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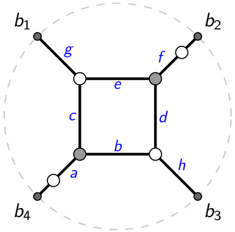

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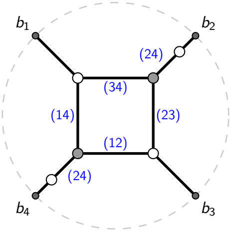

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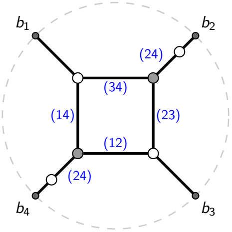

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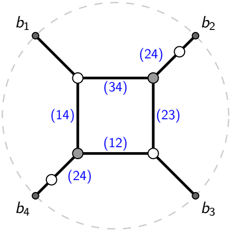

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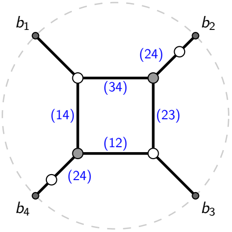

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$$\begin{aligned}
\Delta_{12} &= (23)\cdot(24) \\
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\end{aligned}$$

by Ptolemy's theorem

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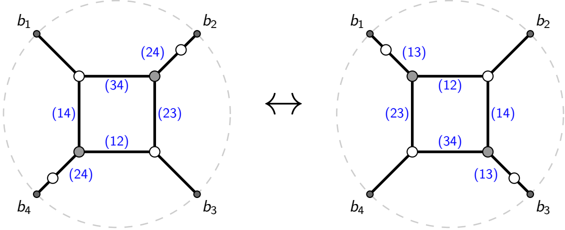

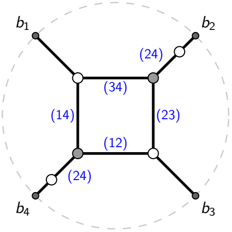

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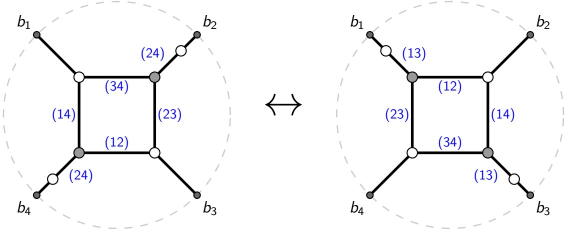
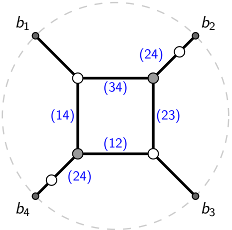


- A graph G is reduced if it has the minimal number of faces among all graphs with the same strand permutation.



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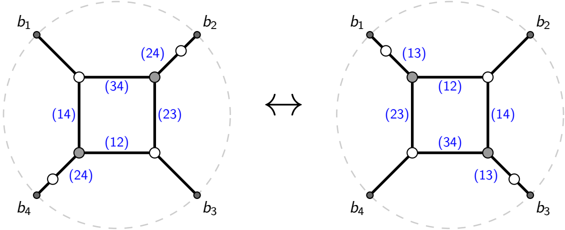

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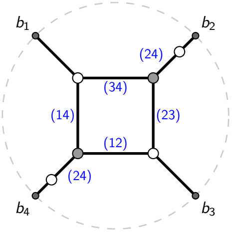

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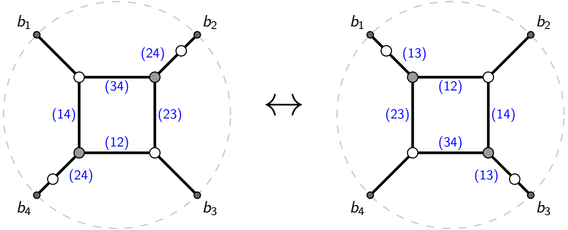

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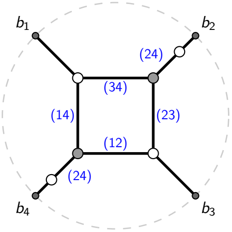
- Conclusion: for each reduced G with $f_G = f$, $\mathrm{Meas}_G(\mathrm{wt}_{\boldsymbol{\theta}}) = \mathrm{Meas}_f(\boldsymbol{\theta})$ depends only on f and $\boldsymbol{\theta}$.



$$(pq) := \sin(\theta_q - \theta_p)$$

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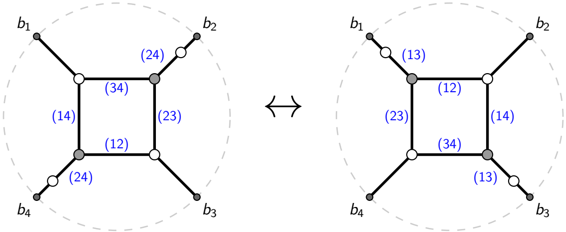
\Delta_{24} &= (14)\cdot(23)+(12)\cdot(34)=(13)\cdot(24)
\end{aligned}$$

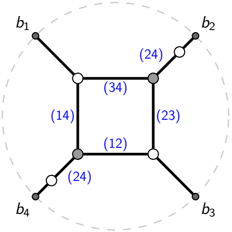
by Ptolemy's theorem

Critical dimer model

- This model is invariant under square moves:



- A graph G is reduced if it has the minimal number of faces among all graphs with the same strand permutation.
- Any two reduced G, G' with $f_G = f_{G'}$ are related by square moves.
- Conclusion: for each reduced G with $f_G = f$, $\mathrm{Meas}_G(\mathrm{wt}_{\boldsymbol{\theta}}) = \mathrm{Meas}_f(\boldsymbol{\theta})$ depends only on f and $\boldsymbol{\theta}$.
- Formula for $\mathrm{Meas}_f(\boldsymbol{\theta})$ in terms of f and $\boldsymbol{\theta}$?



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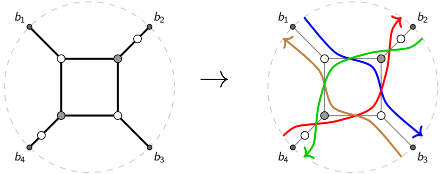
- Strand permutation: $f \in S_n$.



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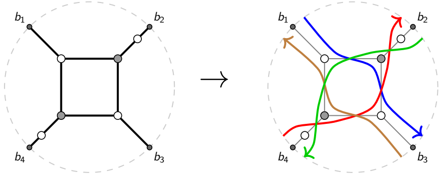

$J_1 = \{2\}$ $J_2 = \{3\}$ $J_3 = \{4\}$ $J_4 = \{1\}$

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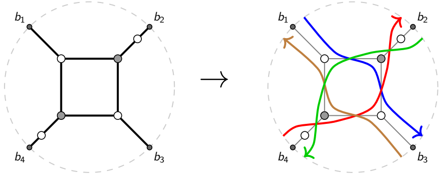

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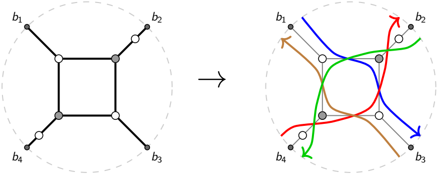

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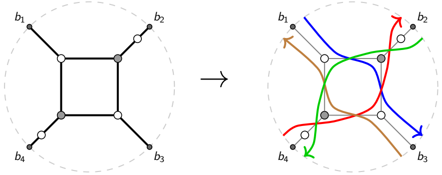

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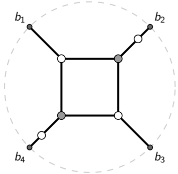
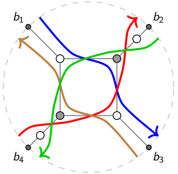


$\to$



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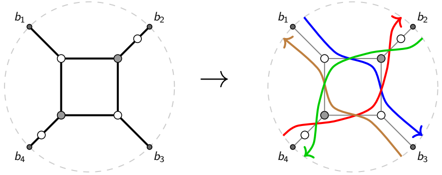

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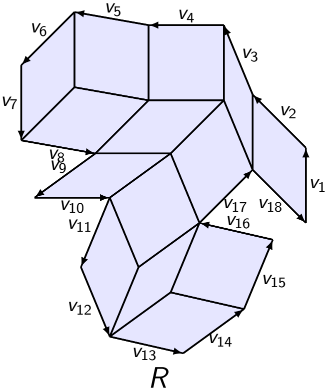

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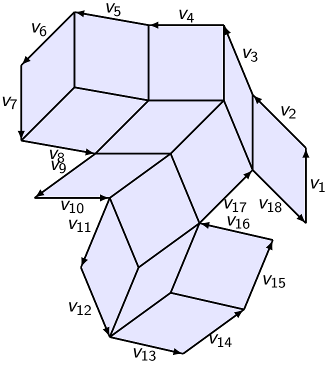
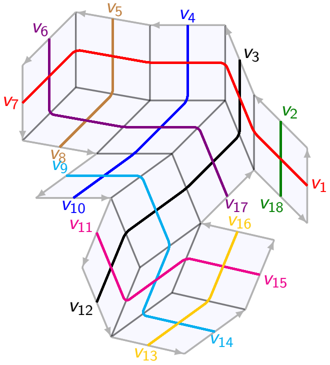


R



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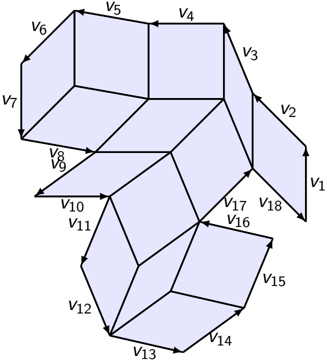

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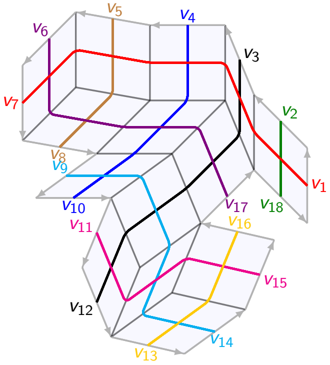

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- This implies the above formulas for regular polygons in the Ising and electrical cases.

Thanks!