Perturbing Isoradial Triangulations

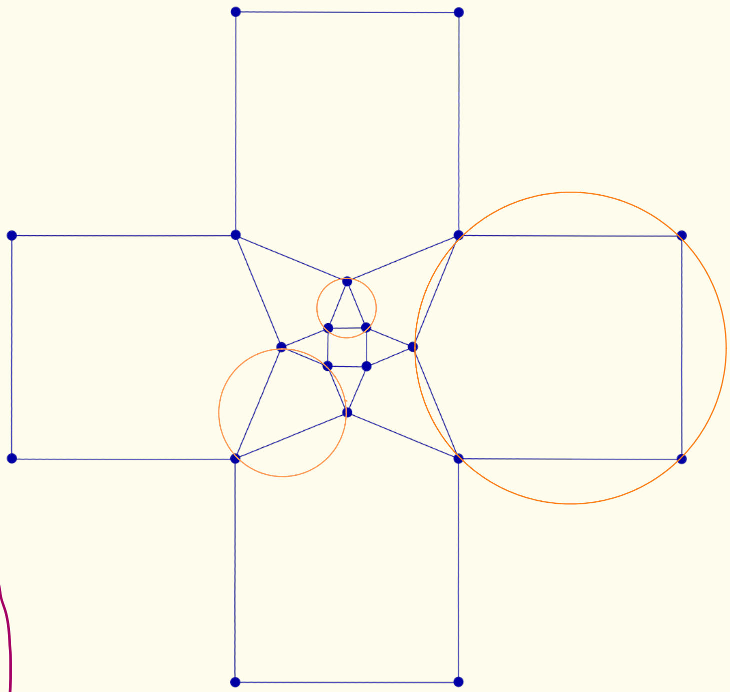
joint work with F. David (IPhT)

(Jussieu 2021)

(infinite) Delaunay Graph G

- planar graph with vertex embedding $z : V(G) \to \mathbb{C}$
- edges $\mapsto$ straight line segments
  faces $\mapsto$ cyclic polygons
- $\forall v \in V(G)$ and $\forall X \in F(G)$
  $z(v) \notin$ int of circumdisk of X

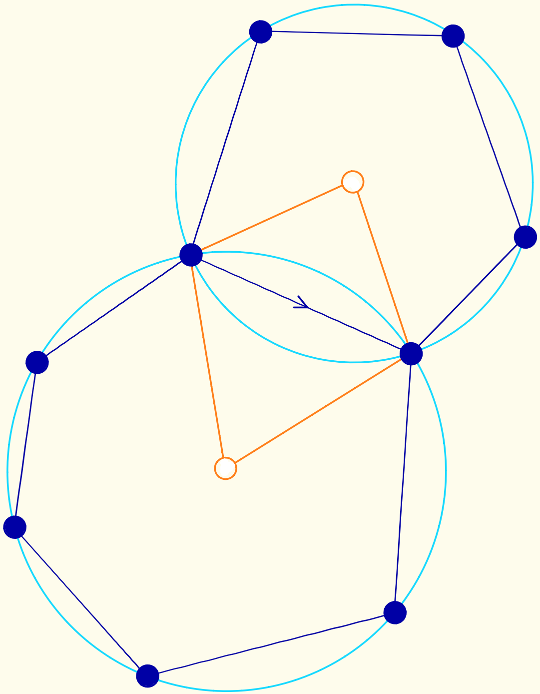
$$\left(\begin{array}{l} \overline{uv} \in E(G) \text{ iff } \exists w \neq u,v \text{ vertex} \\ \text{such that } \forall w' \neq u,v,w \text{ vertex} \\ \operatorname{im}\left[z(u), z(v) ; z(w), z(w') \right] \in \mathbb{R}_{>0} \\ \qquad \text{(cross-ratio)} \end{array} \right)$$



$V(G)$ vertex set

$E(G)$ edge set

$F(G)$ face set



Notation / set-up :

for $u \in V(G)$, $z(u) \in \mathbb{C}$ coordinate

for an oriented edge $\vec{uv}$ have

- north and south faces X_n, X_s
- circumcenters x_n, x_s
- circumradii R_n, R_s
- angles θ_n, θ_s
- conformal angle

$$\theta(\overline{uv}) = \frac{1}{2}\left(\theta_n(\vec{uv}) + \theta_s(\vec{uv})\right)$$

$$\mathbb{C}^{V(G)} = \{\phi : V(G) \to \mathbb{C}\}$$

$$\mathbb{C}^{F(G)} = \{\phi : F(G) \to \mathbb{C}\}$$

vector spaces complex-valued functions

David-Eynard Kähler operator :

$$\mathcal{D} : \mathbb{C}^{V(G)} \longrightarrow \mathbb{C}^{V(G)}, \quad \phi \in \mathbb{C}^{V(G)}$$

$$\mathcal{D}\phi(u) = \sum_{\substack{\text{edges} \\ u \to v}} \omega_{\mathcal{D}}(\vec{uv})\left[\phi(u) - \phi(v)\right]$$

$$\omega_{\mathcal{D}}(\vec{uv}) = \frac{1}{2}\left[\frac{\tan\theta_n(\vec{uv}) + i}{R_n^2} + \frac{\tan\theta_s(\vec{uv}) - i}{R_s^2}\right]$$

Remarks :

(1) $\overline{\mathcal{D}}^T = \mathcal{D}$ (hermitian)

(2) $\mathcal{D}\mathbb{1} = \mathcal{D}z = \mathcal{D}z^2 = 0$
(three zero modes)

(3) Special Case :

if in addition G is isoradial

(i.e. $\exists R_{cr} > 0$ such that circumradius $R(f) = R_{cr}$ for each face $f \in F(G)$)

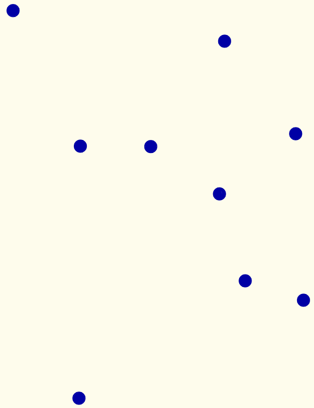
then $\theta_n(\vec{uv}) = \theta_s(\vec{uv})$ and so

$$\omega_{\mathcal{D}}(\vec{uv}) = R_{cr}^{-2} \tan\theta(\overline{uv})$$

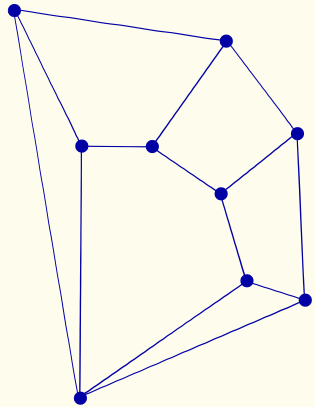
i.e. $\mathcal{D} = R_{cr}^{-2} \Delta_{cr}$

Kenyon's critical Laplacian

voronoi construction



finite configuration of N points in $\mathbb{CP}^1$ $\Big\} \in \mathcal{M}_{0,N}$



associated Delaunay graph ; equipped Kähler operator

$\downarrow$ $SL_2(\mathbb{C})$-invariant

Get measure on

$$\mathcal{M}_{0,N} \subset \underbrace{\mathbb{CP}^1 \times \dots \times \mathbb{CP}^1}_{N \text{ times}}$$

$$\frac{dz_1^2 \cdots dz_N^2 \det \mathcal{D}_{[4,N]}}{|z_1 - z_2|^2 \cdot |z_2 - z_3|^2 \cdot |z_1 - z_3|^2}$$

= pullback of the Weil-Petersson measure

(Charbonnier - David-Eynard)

$\mathcal{M}_{0,N} \simeq$ space of Delaunay graphs on $\mathbb{CP}^1$ with N labeled vertices $\circlearrowleft$ $SL_2(\mathbb{C})$-action by möbius trans.

$SL_2(\mathbb{C})$-equivariant isomorphism.

Basic Problem: study how $\det' \mathcal{D}$ varies as the Delaunay graph G is perturbed/deformed.

Start with an infinite Delaunay graph G_{cr} which is isoradial and deform coordinate embedding z_{cr}

$$z_{\underline{\epsilon};\ell}(v) := z_{cr}(v) + \epsilon_1 F_{1;\ell}(v) + \cdots + \epsilon_m F_{m;\ell}(v)$$

$\epsilon_1, \ldots, \epsilon_m > 0$ independent deformation parameters

$\ell > 0$ scaling parameter

$F_1, \ldots, F_m : \mathbb{C} \to \mathbb{C}$ smooth, compact supports

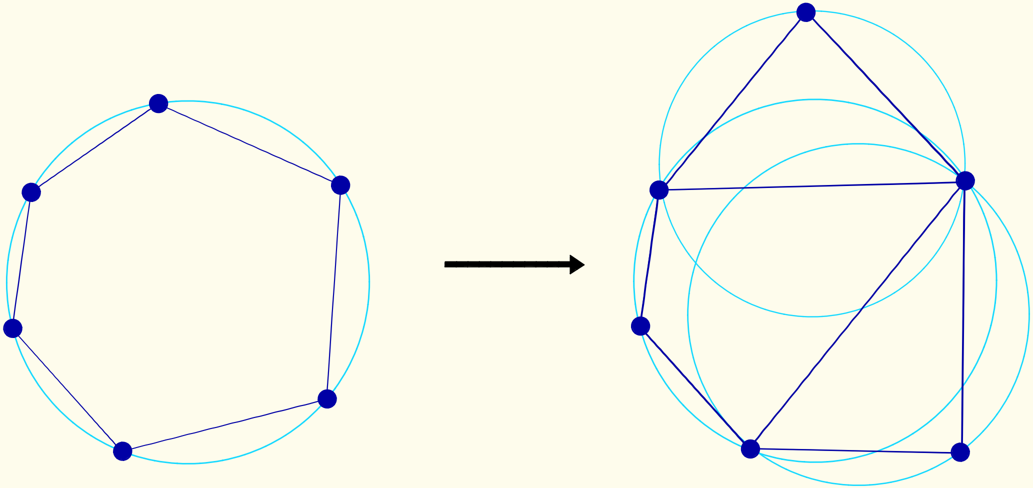
$\operatorname{supp} F_i \cap \operatorname{supp} F_j = \emptyset$ for all i, j

$F_{i;\ell}(v) := \ell \cdot F_i\left(z_{cr}(v)/\ell\right)$

$\boxed{\text{We get Delaunay graph } G_{\underline{\epsilon};\ell} \text{ vertex set } V(G_{\underline{\epsilon};\ell}) = V(G_{cr}) \text{ with Kähler operator } \mathcal{D}_{\underline{\epsilon},\ell}}$

We want to compare $\mathcal{D}_{\underline{\epsilon};\ell}$ on $G_{\underline{\epsilon};\ell}$ and Δ_{cr} on G_{cr}

<u>Difficulty</u>: Delaunay constaints create/remove edges and so $E(G_{\underline{\epsilon};\ell}) \neq E(G_{cr})$ and $F(G_{\underline{\epsilon};\ell}) \neq F(G_{cr})$



Z_{cr} embedding

$Z_{\underline{\epsilon};\ell}$ embedding

1st approach: find bounds $\hat{\epsilon}_i$ (depending on F_i) such that

stable incidence relations within the germ of the deformation

$$\begin{cases} ① \quad E(G_{cr}) \subseteq E(G_{\underline{\epsilon};\ell}) \quad \text{whenever } 0 \leq \epsilon_i < \hat{\epsilon}_i \\ ② \quad E(G_{\underline{\epsilon};\ell}) = E(G_{\underline{\epsilon}';\ell}) \quad \text{whenever } 0 < \epsilon_i, \epsilon_i' < \hat{\epsilon}_i \end{cases}$$

This allows us to define a limit graph $G_{0^+;\ell}$ (depending on $F_1, \dots, F_m$)

vertex set $V(G_{0^+;\ell}) = V(G_{cr})$

edge set $E(G_{0^+;\ell}) \supseteq E(G_{cr})$

and embedding $z_{0^+;\ell} = z_{cr}$

in general $G_{0^+;\ell}$ is isoradial and "weakly" Delaunay

Note: if G_{cr} is a triangulation then so is $G_{\underline{\epsilon};\ell}$

furthermore $E(G_{\underline{\epsilon};\ell}) = E(G_{cr})$

$F(G_{\underline{\epsilon};\ell}) = F(G_{cr})$

therefore $G_{0^+;\ell} = G_{cr}$.

2nd approach: do not require that $E(G_{cr}) \subseteq E(G_{\underline{\epsilon};\ell})$ but bound deformation parameters so that the growth of $R_{\underline{\epsilon}}(x)$ is bounded as $\underline{\epsilon}$ varies

$\rightarrow$ allows us to estimate $\delta\mathcal{D} = \int_0^{\epsilon} d\varepsilon\, \mathcal{D}'(\varepsilon)$

Proposition: $(m=1)$ let $M_1 = \sup_{z\in\mathbb{C}} \max\left\{ |\partial F(z)|, |\bar\partial F(z)| \right\}$

$$M_2 = \sup_{z\in\mathbb{C}} \max\left\{ |\partial^2 F(z)|, |\partial\bar\partial F(z)|, |\bar\partial^2 F(z)| \right\}$$

$$\epsilon_{max} = \frac{1}{2M_1}\left(1 - \left(1 + \frac{M_1}{R_{cr} M_2}\right)^{-1/4}\right)$$

then $\bar{R}_-(\epsilon) \leq R_\epsilon(x) \leq \bar{R}_+(\epsilon) \quad \forall x \in F(G_{cr}) \;\; \forall\, 0 \leq \epsilon < \epsilon_{max}$

$$\bar{R}_-(\epsilon) = \frac{R_{cr}(1-2M_1\epsilon)^2}{1 + \frac{8M_2R_{cr}}{M_1}\log\left(\frac{1}{1-2M_1\epsilon}\right)}$$

$$\bar{R}_+(\epsilon) = \frac{R_{cr}}{\left(1+\frac{M_2R_{cr}}{M_1}\right)(1-2M_1\epsilon)^2 - \frac{M_2R_{cr}}{M_1}(1-2M_1\epsilon)^{-2}}$$

set $\delta\mathcal{D} = \mathcal{D}_{\epsilon,\ell} - R_{cr}^{-2}\Delta_{cr}$ and then formally expand

$$\underbrace{\log\det\mathcal{D} - \log\det\Delta_{cr}}_{\text{variation } \delta\log\det\mathcal{D}} = \underbrace{R_{cr}^{2}\,\mathrm{tr}\left[\delta\mathcal{D}\cdot\Delta_{cr}^{-1}\right] - \frac{1}{2}R_{cr}^{4}\,\mathrm{tr}\left[\delta\mathcal{D}\cdot\Delta_{cr}^{-1}\right]^{2} + \cdots}_{\text{each trace term} < \infty}$$

where Δ_{cr}^{-1} is the Green's function of Δ_{cr} characterised by

(1) $\Delta_{cr}\Delta_{cr}^{-1} = \mathbb{1}$ (means that $u \mapsto [\Delta_{cr}^{-1}]_{uv}$ is harmonic)

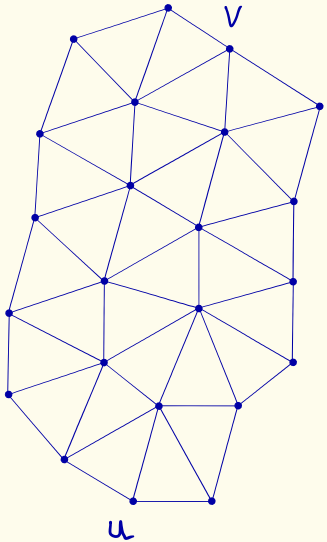
(2) $[\Delta_{cr}^{-1}]_{uv} = O(\log|z_u - z_v|)$ for $u,v \in V(T)$, $|z_u - z_v| \gg 0$

(3) $[\Delta_{cr}^{-1}]_{u,u} = 0$

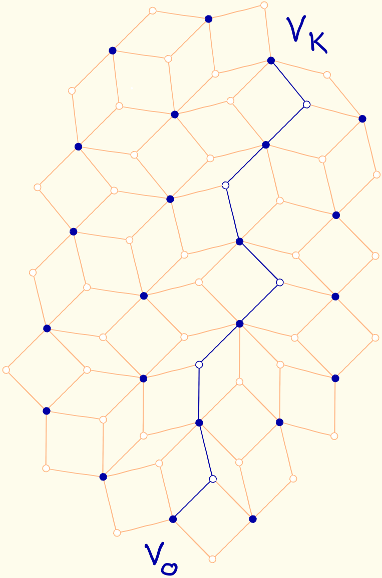
Asymptotics:

$$[\Delta_{cr}^{-1}]_{uv} = -\frac{1}{2\pi}\left(\underbrace{\log 2D + \gamma_{euler}}_{\text{Kenyon 2002}} + \underbrace{\frac{\mathrm{re}[P_3(u,v)]}{6D^3}}_{\text{refinement DS 2021}} + O(D^{-4})\right)$$

$$\boxed{\begin{gathered}\text{here } D = |z_{cr}(u) - z_{cr}(v)| \\ \text{and } |P_3(u,v)| \leq 3D\end{gathered}}$$



isoradial Delaunay graph G_{cr}



bipartite rhombic graph $G^{\diamond}_{cr}$

$v_0, \dots, v_K$ path in $G^{\diamond}_{cr}$ with $v_0 = u$ and $v_K = v$

$$R_{cr} e^{i\alpha_j} = z_{cr}(v_j) - z_{cr}(v_{j-1})$$

$P_3(u,v) :=$

$$R_{cr}\left(e^{3i\alpha_1} + \dots + e^{3i\alpha_K}\right)$$

independent of path

For simplicity assume G_{cr} is a <u>triangulation</u> then

<u>Thm</u> (D, S) let $m=2$ (i.e. bilocal perturbation)

then the $\epsilon_1\epsilon_2$-crossterm of $R_{cr}^4 \operatorname{tr}\left[\delta\mathcal{D}\cdot\Delta_{cr}^{-1}\right]^2$ is

$$\frac{-2}{\pi^2}\sum_{\substack{\text{triangles}\\(X_1,X_2)}} A(X_1)A(X_2)\,\mathrm{re}\left[\frac{\bar\nabla F_{1;\ell}(X_1)\,\bar\nabla F_{2;\ell}(X_2)}{(x_1-x_2)^4}\right] + O\left(D_\ell^{-5}\right)$$

scaling limit

$\ell\to\infty$

$$\boxed{\frac{-2}{\pi^2}\iint_{\Omega_1\times\Omega_2} dx_1^2\,dx_2^2\,\mathrm{re}\left[\frac{\bar\partial F_1(x_1)\,\bar\partial F_2(x_2)}{(x_1-x_2)^4}\right]}$$

where x_i is the center of X_i

$D_\ell = \mathrm{dist}\left(\Omega_1(\ell),\Omega_2(\ell)\right)$ and

$\Omega_i(\ell) = \left\{ z\in\mathbb{C} \,\middle|\, z/\ell \in \operatorname{supp} F_i \right\}$

$\nabla, \bar\nabla : \mathbb{C}^{V(G)} \longrightarrow \mathbb{C}^{F(G)}$

discrete derivatives

$$\nabla\phi(X) = \frac{-1}{4i\,A(X)}\det\begin{pmatrix} 1 & 1 & 1 \\ \bar z(v_1) & \bar z(v_2) & \bar z(v_3) \\ \phi(v_1) & \phi(v_2) & \phi(v_3)\end{pmatrix}$$

v_1, v_2, v_3 vertices of X in counter-clockwise order

Proof involves:

- asymptotic expansion $[\Delta_{cr}^{-1}]_{uv} = -\frac{1}{2\pi}\left(\log 2D + \gamma_{euler} + \frac{re[P_3(u,v)]}{6D^3} + O(D^{-4})\right)$
- operator factorisation $\mathcal{D} = 4\bar{\nabla}^T A R^{-2} \nabla$ where
  $A, R : \mathbb{C}^{F(G)} \to \mathbb{C}^{F(G)}$ diagonal operators $A\phi(X) = A(X)\phi(X)$, $R\phi(X) = R(X)\phi(X)$

  note: Beltrami-Laplace operator $\Delta = 4\, re[\bar{\nabla}^T A \nabla]$

discrete stress energy tensor $T_{\mathcal{D}}$:

$$R_{cr}^2 \, tr[\delta\mathcal{D}\cdot\Delta_{cr}^{-1}] = \begin{cases} -4\, tr\left[\bar{\nabla}^T A\left(\nabla F + \bar{\nabla}\bar{F} + c\bar{\nabla}F + \bar{c}\nabla\bar{F}\right)\nabla\cdot\Delta_{cr}^{-1}\right] & (m=1) \\ -4\, tr\left[\nabla^T A\bar{\nabla}F\nabla\cdot\Delta_{cr}^{-1}\right] - 4\, tr\left[\bar{\nabla}^T A\nabla\bar{F}\,\bar{\nabla}\cdot\Delta_{cr}^{-1}\right] + O(\epsilon^2) \end{cases}$$

$$= -\frac{1}{\pi}\sum_{x\in F(G_{cr})} A(x)\, re\left[\bar{\nabla}F(x)\langle T_{\mathcal{D}}(x)\rangle\right] + \frac{1}{2}\sum_{x\in F(G_{cr})} A(x)\langle T_{\mathcal{D}}^{z\bar{z}}(x)\rangle\, re[\nabla F(x)] + O(\epsilon^2)$$

expectation values

$$T_{\mathscr{D}}(x) = -\frac{\pi}{2} T_{\mathscr{D}}^{\bar{z}\bar{z}}(x) = -4\pi R^{-2}(x)\left[\nabla\phi(x)\nabla\bar{\phi}(x) + C(x)\,\bar{\nabla}\bar{\phi}(x)\bar{\nabla}\phi(x)\right]$$

$$\bar{T}_{\mathscr{D}}(x) = -\frac{\pi}{2} T_{\mathscr{D}}^{zz}(x) = -4\pi R^{-2}(x)\left[\bar{\nabla}\phi(x)\bar{\nabla}\bar{\phi}(x) + \bar{C}(x)\,\bar{\nabla}\phi(x)\nabla\phi(x)\right]$$

$$T_{\mathscr{D}}^{z\bar{z}}(x) = 8R^{-2}(x)\,\bar{\nabla}\phi\,\nabla\bar{\phi}$$

components of the stress energy tensor $\mathbb{T}_{\mathscr{D}}$

where $\langle A\rangle = Z_{\epsilon;\ell}^{-1}\int \mathcal{D}[\phi]\, A[\phi]\, e^{-\bar{\phi}\cdot\mathscr{D}_{\epsilon;\ell}\phi}$

$$Z_{\epsilon;\ell} = \int \mathcal{D}[\phi]\, e^{-\bar{\phi}\cdot\mathscr{D}_{\epsilon;\ell}\phi}$$

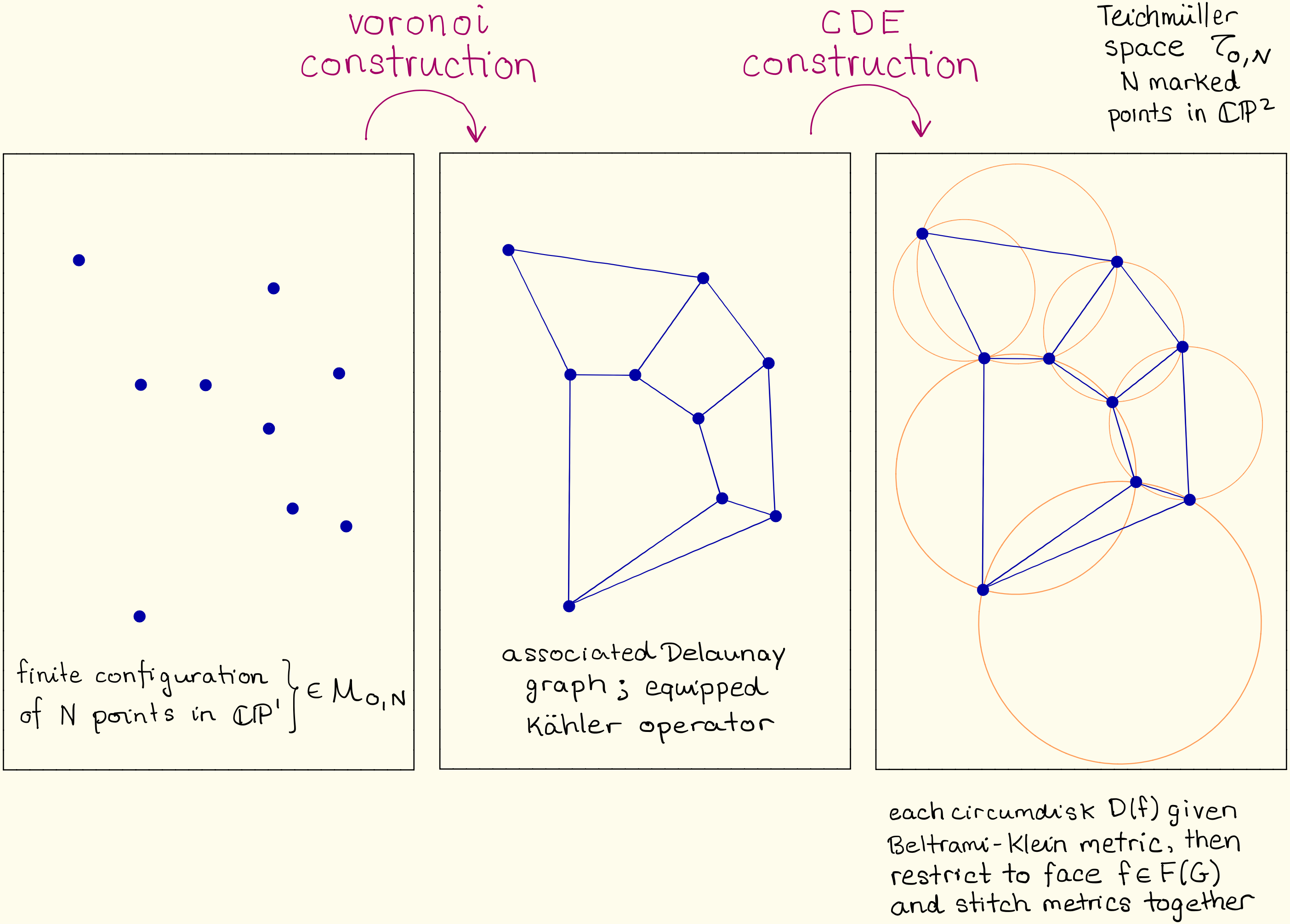
$$= \det^{-\frac{1}{2}}\mathscr{D}_{\epsilon;\ell}$$

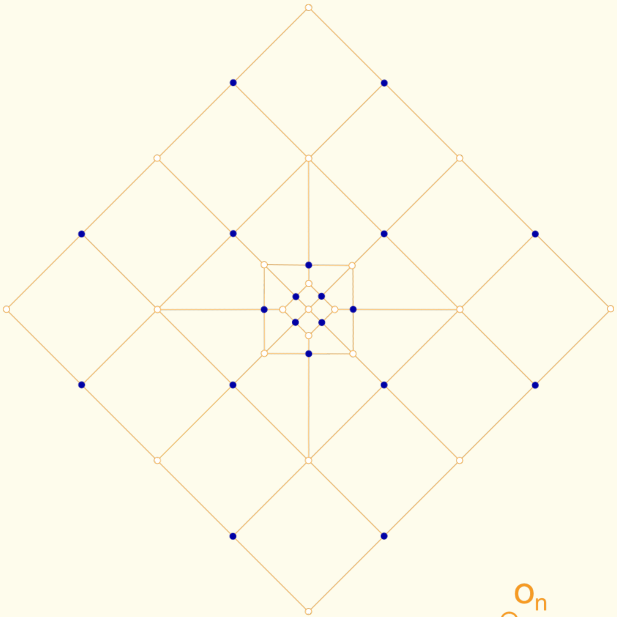
and we formally compute in the language of functional integrals using Wick's theorem:

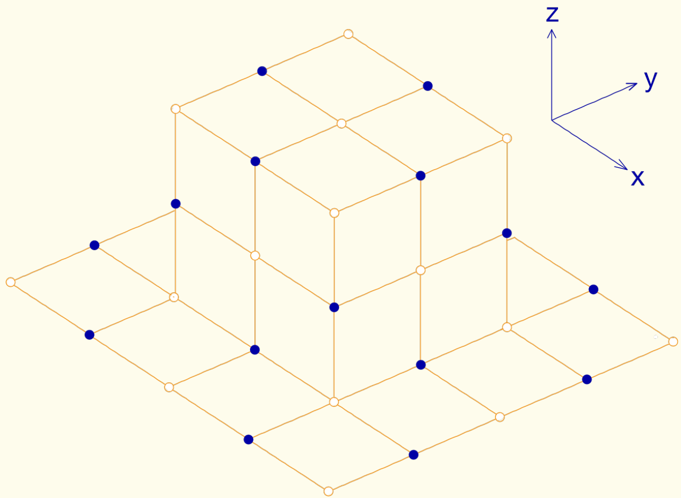
$$\left[\mathscr{D}_{\epsilon;\ell}\right]^{-1}_{uv} = \langle\phi(u)\bar{\phi}(v)\rangle$$

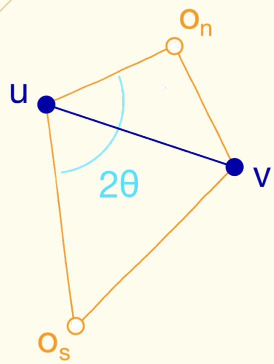
etc...

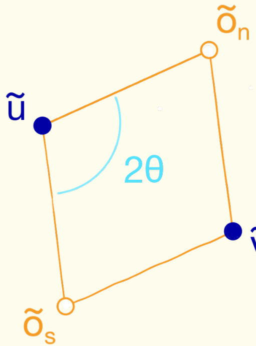
we have a similar formula for $R_{cr}^4\,\mathrm{tr}\left[\delta\mathscr{D}\cdot\Delta_{cr}^{-1}\right]^2$ in terms of components of the stress energy tensor.


voronoi construction
CDE construction
Teichmüller space $\mathcal{T}_{0,N}$
N marked points in $\mathbb{CP}^2$
finite configuration of N points in $\mathbb{CP}^1$ $\} \in \mathcal{M}_{0,N}$
associated Delaunay graph; equipped Kähler operator
each circumdisk D(f) given Beltrami-Klein metric, then restrict to face f ∈ F(G) and stitch metrics together









$$\sum_{e \in \partial x} \theta(e) = \frac{\pi}{2} \qquad \circ$$

"flatness"

$$\sum_{v \in \partial e} \theta(e) = \pi \qquad \bullet$$

always

def a Delaunay triangulation T is isoradial if all circumradii are equal; $R_f = R_{cr} > 0 \ \forall f \in F(T)$

$\boxed{\begin{array}{l} \text{in this case } \underbrace{\Delta = \mathcal{D} = \Delta_{conf}} \\ (\text{and } R_{cr} = 1) \ \text{"critical laplacian" } \Delta_{cr} \end{array}}$

Green's function Δ_{cr}^{-1} (studied by R. Kenyon '02)

characterised by properties

(1) $\Delta_{cr} \Delta_{cr}^{-1} = \mathbb{1}$ (means that $u \longmapsto [\Delta_{cr}^{-1}]_{uv}$ is harmonic)

(2) $[\Delta_{cr}^{-1}]_{uv} = O(\log|z_u - z_v|)$

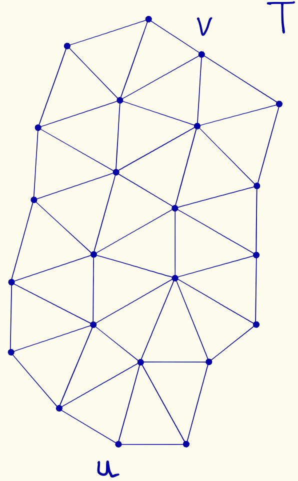
for $u, v \in V(T)$, $|z_u - z_v| >> 0$

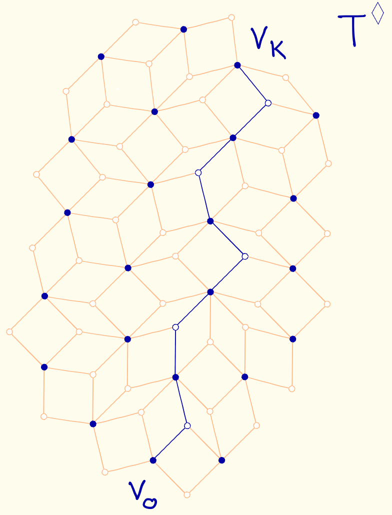
(3) $[\Delta_{cr}^{-1}]_{u,u} = 0$

by [Kenyon 2002]

(1) $[\Delta_{cr}^{-1}]_{uv} = \frac{1}{2\pi} \mathrm{re} \int_0^1 \frac{dt}{t} \left(E_{\underline{\theta}}(-t) - 1 \right)$

where $E_{\underline{\theta}}(z) = \prod_{j=1}^{K} \frac{z + e^{i\theta_j}}{z - e^{i\theta_j}}$





$v_0 = u$ and $v_K = v$

$e^{i\theta_j} = z(v_j) - z(v_{j-1})$

rhombic graph

$v_0, \dots, v_K$ path